Solving Graph Coloring Problems with Abstraction and Symmetry

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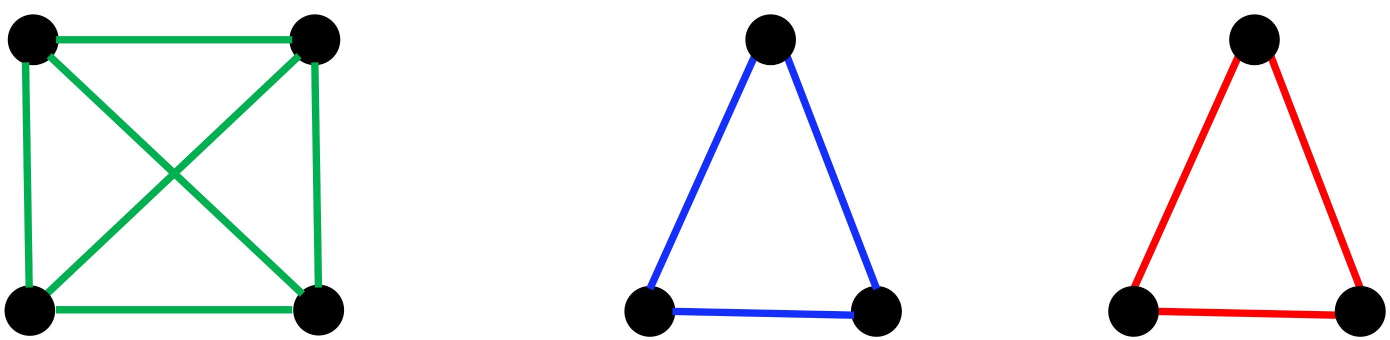
Main Problem

- Is it possible to color the edges of the complete graph on 30 vertices, with 3 colors (green,blue and red) and avoid K_4, K_3, K_3 ?



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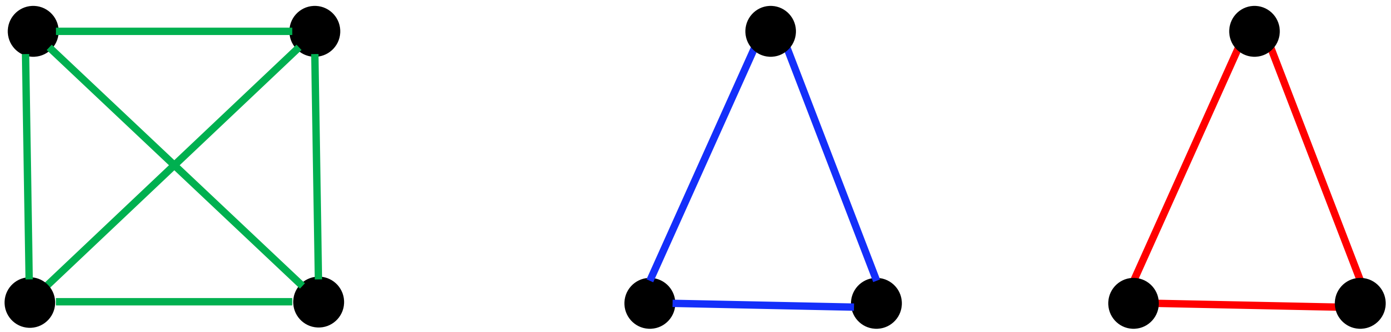


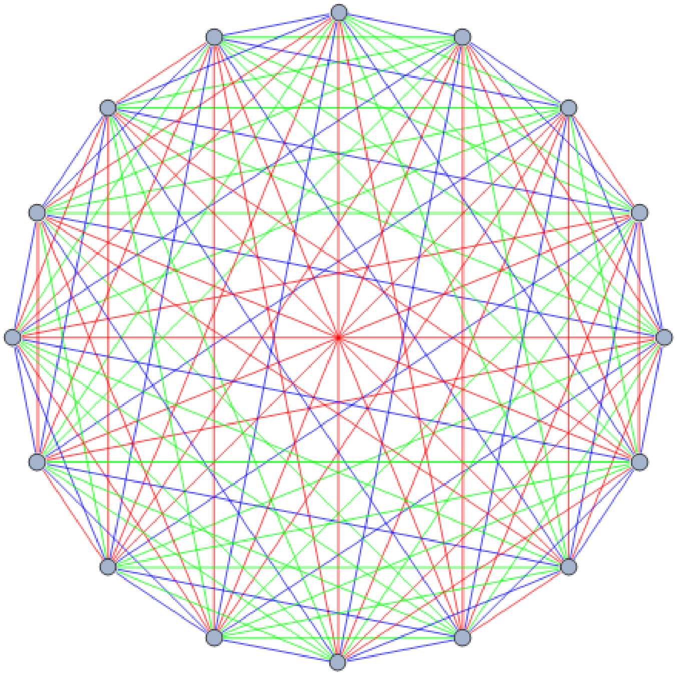
Solution for this problem = Solving a 50 years open instance

Multicolor Ramsey

- The multicolor Ramsey number $R(r_1, r_2, \ldots, r_k)$ is the size of the minimal complete graph such that every edge-coloring with k colors contains a monochromatic complete sub-graph K_{r_i} for some color i.

$|R(3,3,3;16)| = 2$

$R(3,3,3) = 17$



R(3,3,3;16) coloring
(No K_3, K_3, K_3 subgraphs)

Ramsey is Hard

- Determining a Ramsey number is hard [Burr 1986]
- Common approach: using CSP/SAT.
  - We used BEE. An encoder/compiler from constraints to CNF using Minisat as underlying sat solver
- The open instances remain open

Our Approach

- $R(4,3,3)$ was conjectured to be the smallest Ramsey instance that will be solved "soon"

  $30 \leq R(4,3,3) \leq 31$ [K. Piwakowski and S. Radzisowsky 2001]

- We focus on searching $R(4,3,3;30)$ coloring:
  - If there exists a coloring then $R(4,3,3) = 31$
  - Otherwise $R(4,3,3) = 30$

Outline

- Introduction & Motivation
- Embedding
- SAT/CSP Encoding
- The Method
- Results

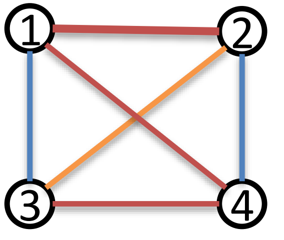
Notation

- In our context:
  - Graph = complete graph K_n
  - Coloring = edge coloring

Adjacency matrix

$$\begin{bmatrix} 0 & 2 & 3 & 2 \\ 2 & 0 & 1 & 3 \\ 3 & 1 & 0 & 2 \\ 2 & 3 & 2 & 0 \end{bmatrix}$$

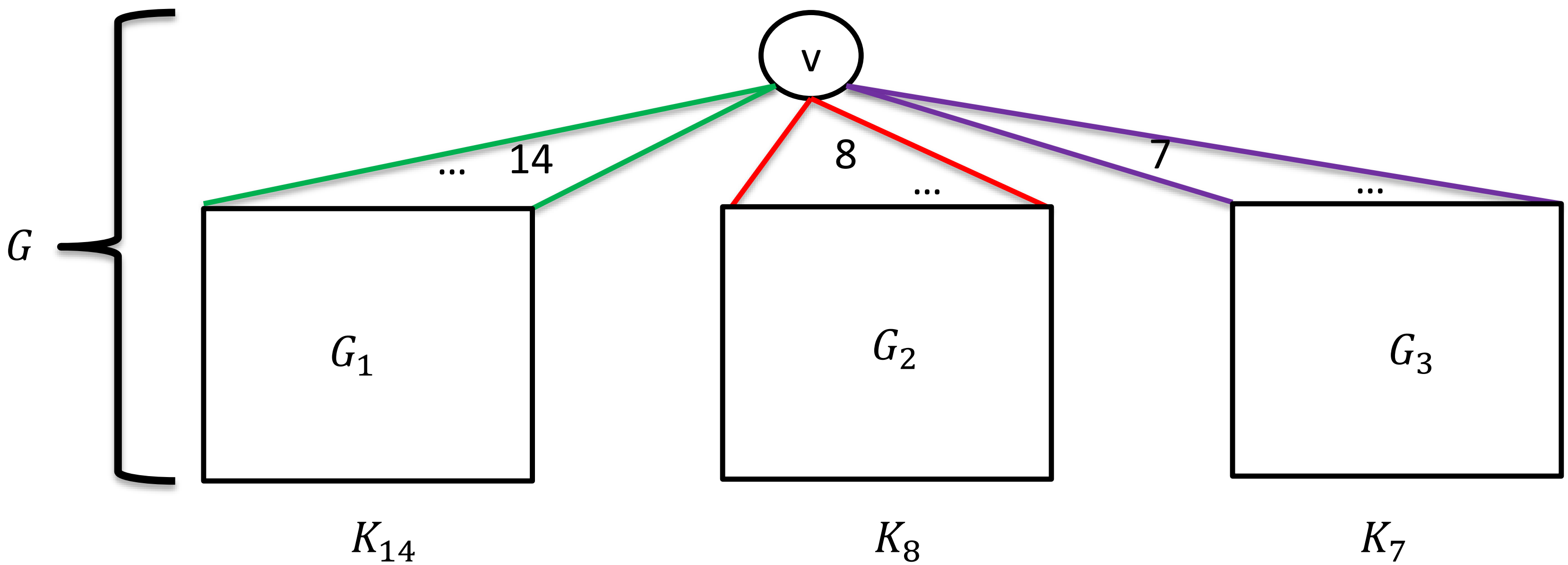
graph



Color degree tuple of vertex 1 = (0,2,1)

Embedding

- Let G be a $R(4,3,3; 30)$ coloring
- Assume $v \in V(G)$ has color degrees (14,8,7)



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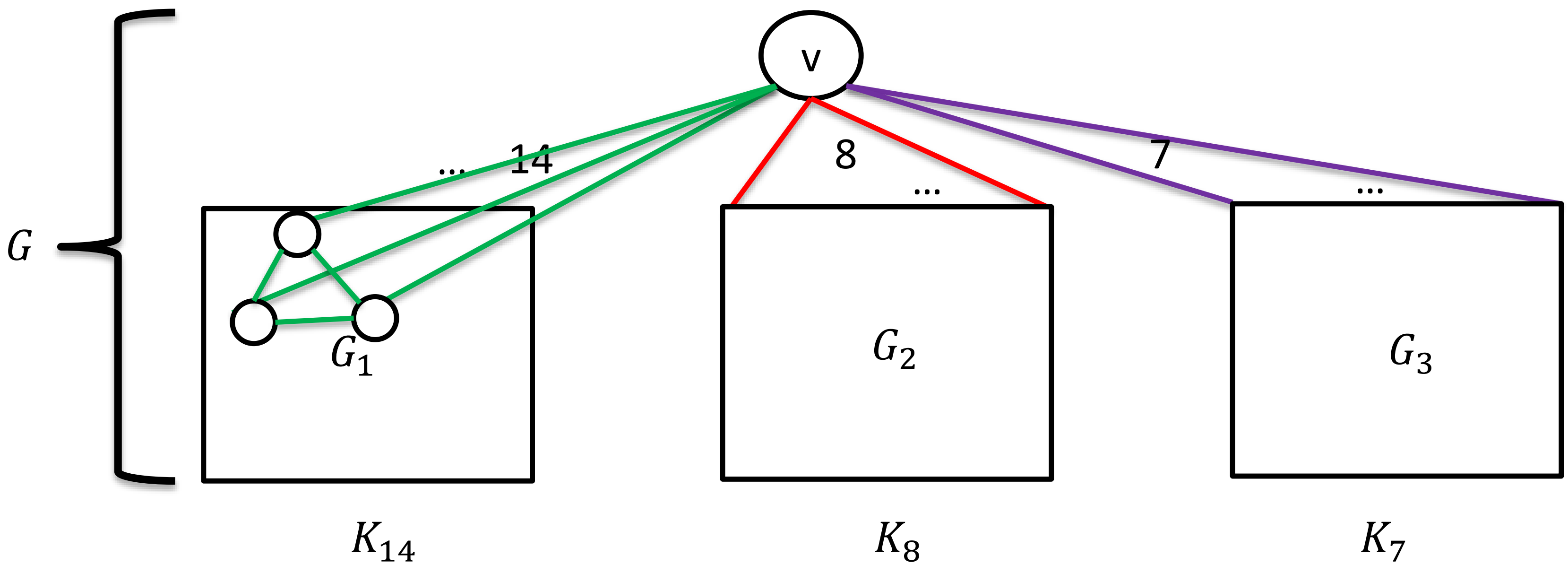

- G_1 contains green K_3 ➔ G contains green K_4

Embedding

- Let G be a $R(4,3,3;30)$ coloring
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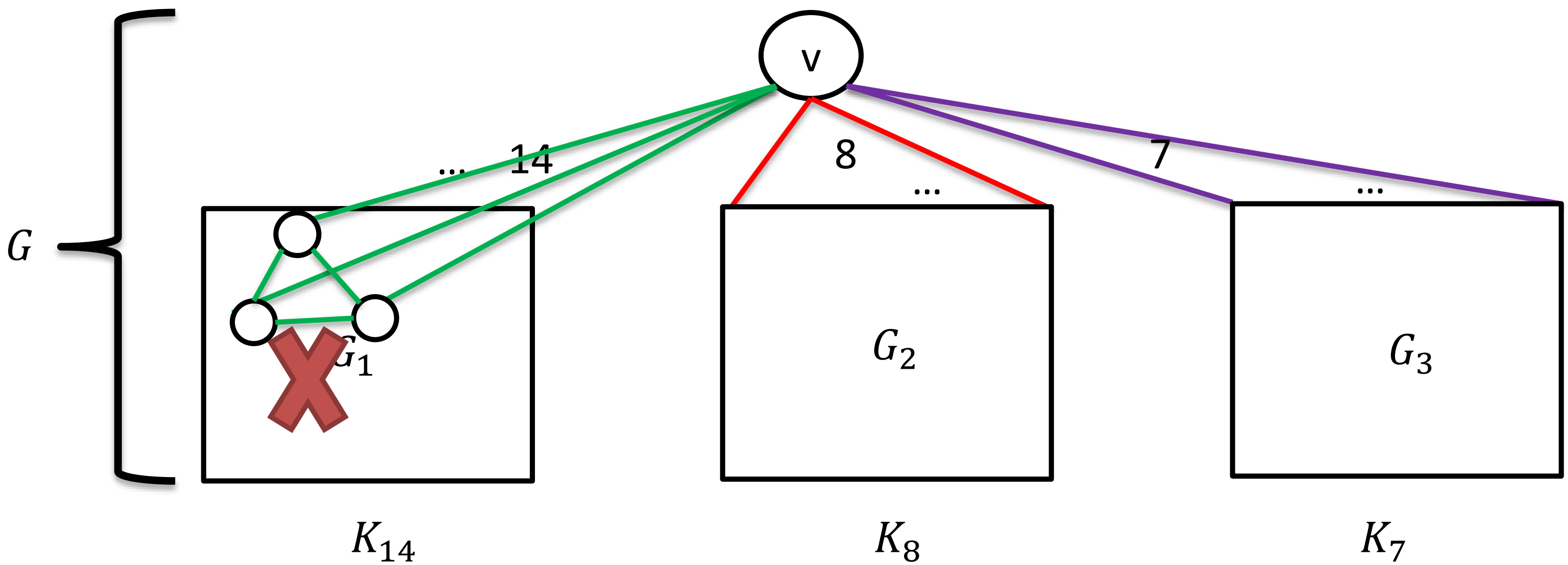

Observation : $G \in R(4,3,3;30) \Rightarrow G_1 \in R(3,3,3;14)$

Embedding

- Let G be a $R(4,3,3;30)$ coloring
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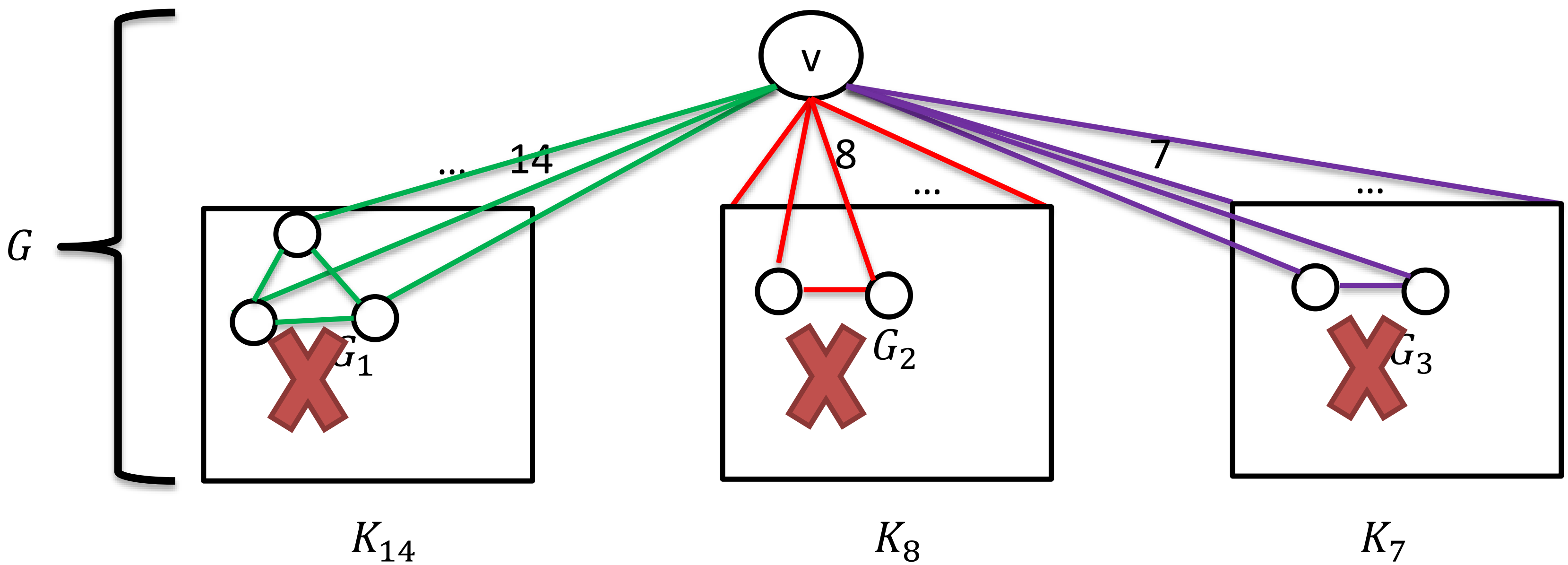
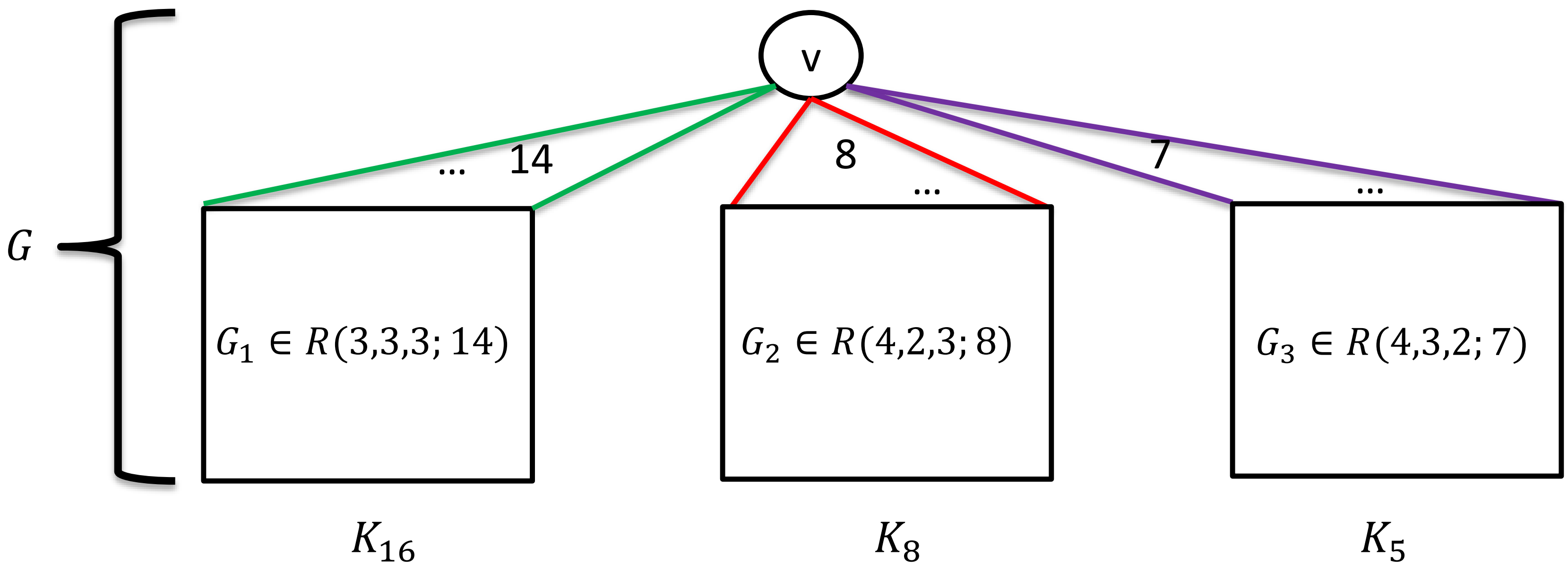


- Similarly, G_2 contains red K_2 ➔ G contains red K_3
  G_3 contains pink K_2 ➔ G contains pink K_3

Embedding

- Let G be a $R(4,3,3;30)$ coloring
- Assume $v \in V(G)$ has color degrees (14,8,7)

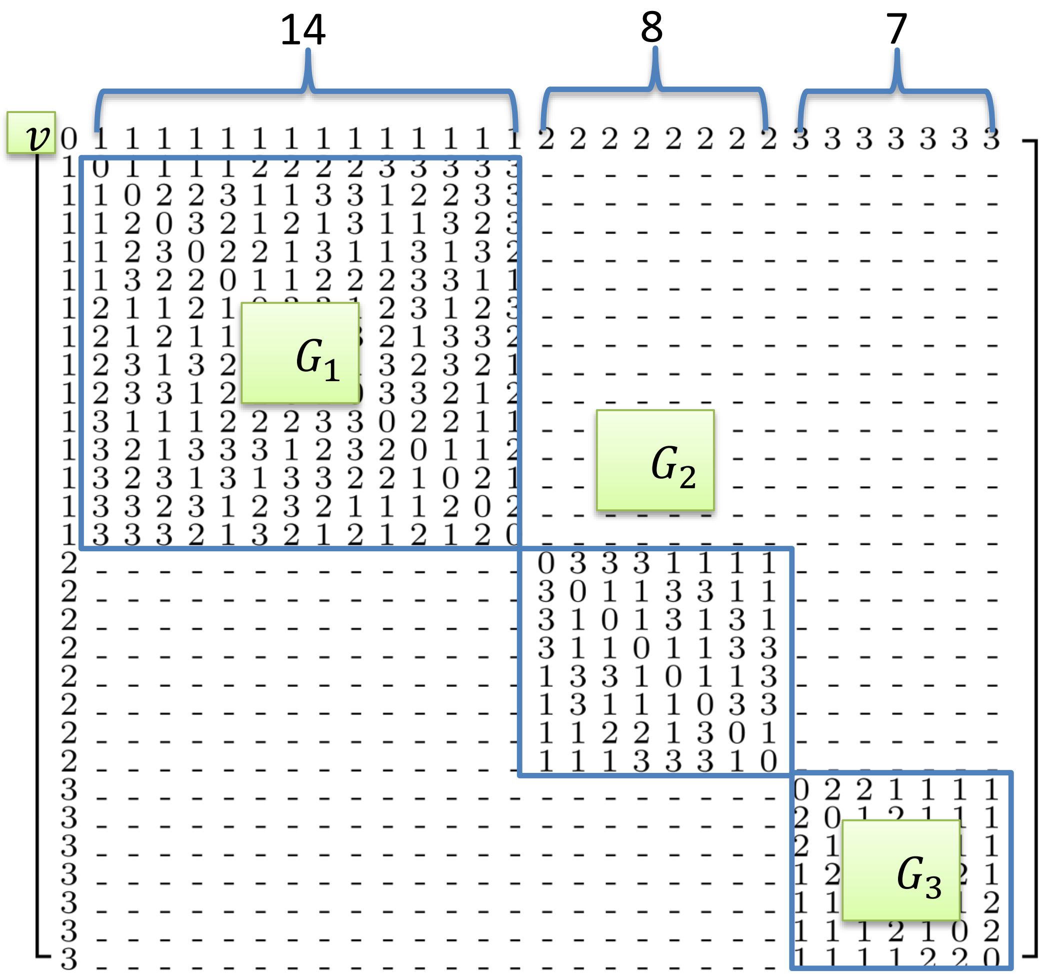


Observation :

$G \in R(4,3,3;30) \Rightarrow G_1 \in R(3,3,3;14), G_2 \in R(4,2,3;8), G_3 \in R(4,3,2;7)$

Embedding

- One of the embedding cases for color degree (14,8,7):



$|R(3,3,3;14)| = 115$

$\times$

$|R(4,2,3;8)| = 3$

$\times$

$|R(4,3,2;7)| = 9$

3105 instances

Embedding - Summarize

- Reduces the search space
- Solve different cases in parallel

Outline

- Introduction & Motivation ✓
- Embedding ✓
- SAT/CSP Encoding
- The Method
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SAT/CSP Encoding

- Represent the coloring A as adjacency matrix
  - $a_{i,j} = color\ of\ edge\ (i,j)$

$$\begin{bmatrix} 0 & & a_{i,j} \\ & \ddots & \\ & & 0 \end{bmatrix}$$

$$\varphi_{adj}^{n,k}(A) = \bigwedge_{1 \le q < r \le n} \left(\underbrace{1 \le A_{q,r} \le k}_{\text{K colors}} \wedge \underbrace{A_{q,r} = A_{r,q}}_{\text{undirected}} \wedge \underbrace{A_{q,q} = 0}_{\text{No self loops}} \right)$$

SAT/CSP Encoding (Cont'd)

- No monochromatic K_3 subgraph

$$\varphi_{K_3}^{n,c}(A) \quad = \bigwedge_{1 \leq q < r < s \leq n} \neg \left(A_{q,r} = A_{q,s} = A_{r,s} = c \right)$$

- No monochromatic K_4 subgraph

$$\varphi_{K_4}^{n,c}(A) \quad = \bigwedge_{1 \leq q < r < s < t \leq n} \neg \left(\; A_{q,r} = A_{q,s} = A_{q,t} = A_{r,s} = A_{r,t} = A_{s,t} = c \; \right)$$

SAT/CSP Encoding (Cont'd)

$$A \in R(4,3,3;30) \Leftrightarrow$$

$$\varphi_{adj}^{30,3}(A) \wedge \varphi_{K_4}^{30,1}(A) \wedge \varphi_{K_3}^{30,2}(A) \wedge \varphi_{K_3}^{30,3}(A)$$

Note: all constraints are straightforward to prescribe as CNF. This is done by BEE

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Known Results for R(4,3,3; 30)

Theorem 1 : $30 \leq R(4,3,3) \leq 31$ and, $R(4,3,3) = 31$ if and only if there exists a $(4,3,3;30)$ coloring κ of K_{30} such that: (1) For every vertex v and $i \in \{2,3\}$, $5 \leq deg_i(v) \leq 8$, and $13 \leq deg_1(v) \leq 16$. (2) Every edge in the third color has at least one endpoint v with $deg_3(v) = 13$. (3) There are at least 25 vertices v for which $deg_1(v) = 13$, $deg_2(v) = deg_3(v) = 8$.

Possible color degrees for a $R(4,3,3;30)$ coloring
(16,8,5)
(16,7,6)
(15,8,6)
(15,7,7)
(14,8,7)
(13,8,8)

[K. Piwakowski and S. Radzisowsky]

Corollary 1 : If G is a $(4,3,3;30)$ coloring, and assuming without loss of generality that the degree of color two is greater equal to the degree of color three, then every vertex in G has degrees in the corresponding colors corresponding to one of the triplets $(13,8,8)$, $(14,8,7)$, $(15,7,7)$, $(15,8,6)$, $(16,7,6)$, $(16,8,5)$.

The Method

- For each possible embedding, we use a SAT solver in an attempt to find a coloring for R(4,3,3;30).
- The solving is done in parallel.

v_1 *degrees*	subgraphs			instances
(16,8,5)	$\lvert R(3,3,3;16)\rvert = 2$	$\lvert R(4,2,3;8)\rvert = 3$	$\lvert R(4,3,2;5)\rvert = 9$	54
(16,7,6)	$\lvert R(3,3,3;16)\rvert = 2$	$\lvert R(4,2,3;7)\rvert = 9$	$\lvert R(4,3,2;6) = 15$	270
(15,8,6)	$\lvert R(3,3,3;15)\rvert = 2$	$\lvert R(4,2,3;8)\rvert = 3$	$\lvert R(4,2,3;6)\rvert = 15$	90
(15,7,7)	$\lvert R(3,3,3;15)\rvert = 2$	$\lvert R(4,2,3;7)\rvert = 9$	$\lvert R(4,2,3;7)\rvert = 9$	162
(14,8,7)	$\lvert R(3,3,3;14)\rvert = 115$	$\lvert R(4,2,3;8)\rvert = 3$	$\lvert R(4,2,3;7)\rvert = 9$	3105
(13,8,8)	$\boldsymbol{\lvert R(3,3,3;13)\rvert = ?}$	$\lvert R(4,2,3;8)\rvert = 3$	$\lvert R(4,2,3;8)\rvert = 3$	?

Results (Part 1)

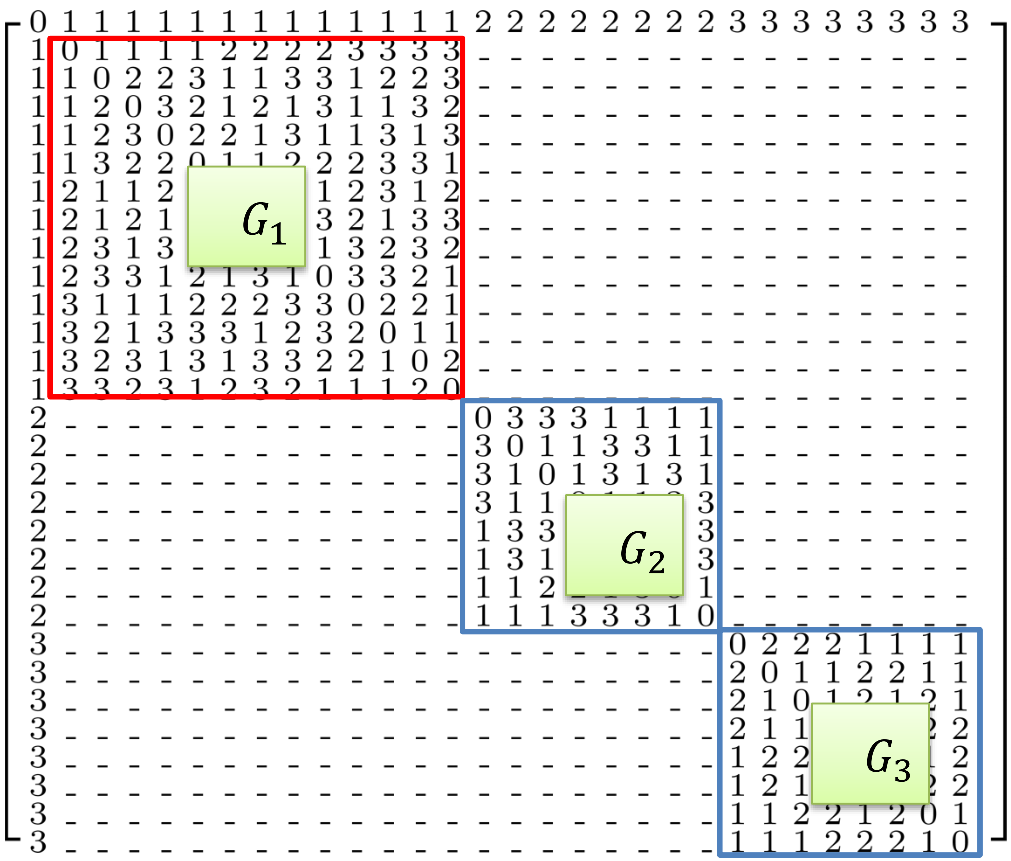
- Solving all cases except (13,8,8)
- All of them are unsatisfiable

v_1 degrees	# instances	# clauses (avg.)	# vars (avg.)	unsat (avg)	unsat (total)
(16,8,5)	54 (2*3*9)	32432	5279	51 sec.	0.77 hrs.
(16,7,6)	270 (2*9*15)	32460	5233	420 sec.	31.50 hrs.
(15,8,6)	90 (2*3*15)	33607	5450	93 sec.	2.32 hrs.
(15,7,7)	162 (2*9*9)	33340	5326	1554 sec.	69.94 hrs.
(14,8,7)	3105 (115*3*9)	34069	5324	294 sec.	253.40 hrs.

Contribution 1 : Any $(4,3,3;30)$ coloring, if one exists, is $(13,8,8)$ regular.

What is Missing to find the exact $R(4,3,3)$ number

(13,8,8) case:



|R(3,3,3;13)| =?

$\times$

|R(4,2,3;8)| =3

$\times$

|R(4,3,2;8)| =3

? instances

Computing R(3,3,3;13) Colorings

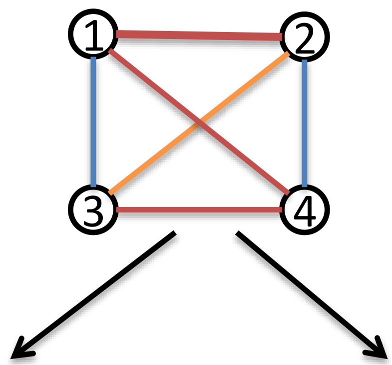
- First try: straightforward encoding with **symmetry** breaking [Codish et al. 2013] – no success

$$ramsey(A) \bigwedge sb_\ell(A)$$

	n	$\#\backslash_\approx$	no sym break			with sym break			
			#vars	#clauses	time	#vars	#clauses	time	#
Unsat	17	0	408	2584	3042.10	4038	20734	0.15	0
	16	2	360	2160	$>$ 24 hrs.	3328	17000	0.14	6
	15	2	315	1785	$>$ 24 hrs.	2707	13745	0.37	66
	14	115	273	1456	$>$ 24 hrs.	2169	10936	259.56	24635
	13	?	234	1170	$>$ 24 hrs.	1708	8540	$>$ 24 hrs	?

Abstraction- Degree Matrix

- Same as degree sequence for colorings



2	1	0
1	1	1
1	1	1
2	1	0

Degree matrix

3
3
3
3

Degree sequence

Results (Part 2) - Computing $R(3,3,3;13)$

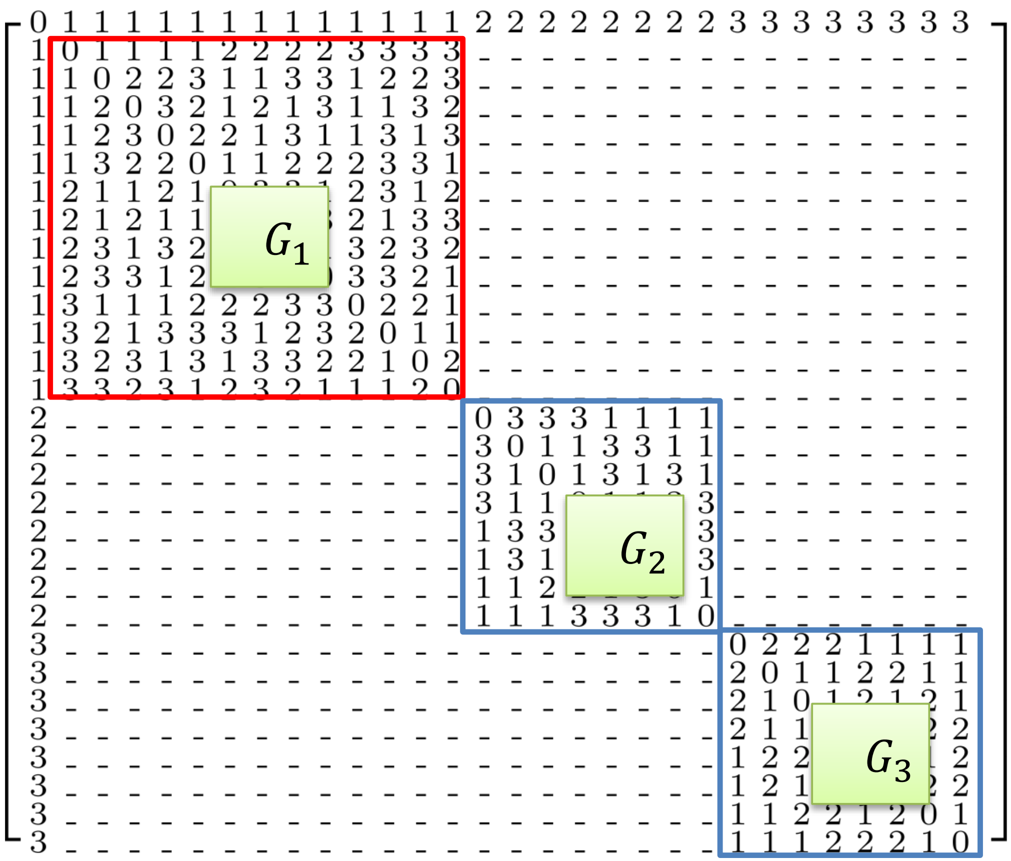
- Find all colorings per degree matrix
- Reduce isomorphic colorings

Step	Notes	Computation Times		
1	compute all Ramsey $(3,3,3;13)$ colorings per degree matrix (999 instances, 129,188 solutions)	MiniSAT total: 308.23 hr. hardest: 9.15 hr.	CryptoMiniSAT total: 136.31 hr. hardest: 4.3 hr.	Glucose total: 373.2 hr. hardest: 17.67 hr.
2	reduce modulo $\approx$. (78,892 solutions)	`nauty`, fast (minutes)		

Contribution 2 : $|R(3,3,3;13)| = 78{,}892$

Completing the search for $R(4,3,3;30)$

(13,8,8) case:



$|R(3,3,3;13)| = 78,892$

$\times$

$|R(4,2,3;8)| = 3$

$\times$

$|R(4,3,2;8)| = 3$

710,028 instances

Completing the search for $R(4,3,3;30)$

- Actually only 78,892 cases

R(4,2,3;8) colorings

$$\begin{matrix} 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 2 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 1 & 0 & 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 & 0 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & 1 & 0 & 2 & 2 \\ 1 & 1 & 2 & 2 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 & 1 & 0 \end{matrix}$$

$$\begin{matrix} 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 & 2 & 0 & 1 & 1 \\ 1 & 1 & 2 & 1 & 2 & 1 & 0 & 2 \\ 1 & 1 & 1 & 2 & 1 & 1 & 2 & 0 \end{matrix}$$

$$\begin{matrix} 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 & 2 & 0 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 2 & 1 & 1 & 2 & 0 \end{matrix}$$

approximation

$$\begin{matrix} 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 2 & A & 1 & 1 \\ 2 & 1 & 0 & 1 & A & B & 2 & 1 \\ 2 & 1 & 1 & 0 & 1 & B & A & 2 \\ 1 & 2 & A & 1 & 0 & B & C & A \\ 1 & A & B & B & B & 0 & A & A \\ 1 & 1 & 2 & A & C & A & 0 & B \\ 1 & 1 & 1 & 2 & A & A & B & 0 \end{matrix}$$

$+$

$constraints$:

1) $A, B, C \in \{1,2\}$

2) $A \neq B$

Results for (13,8,8) case

- 78,890 from 78892 finished
- All unsatisfiable

time (hrs)	# instances	% instances (Δ)
10	56,363	71.445 %
20	65,914	12.107 %
100	77,263	14.386 %
500	78,791	1.936 %
1000	78,869	0.1 %
1800	78,890	0.026 %

Time required per instance

Discussion

- Proofs based on UNSAT results present a verification problem
- Gaining confidence in the results
  - Apply the software on satisfiable instances
  - We independently reproduce unpublished results

Positive Example

- A R(4,3,3;29) coloring has been observed [Kalbfleisch 1966]

$$\begin{bmatrix}
0&1&1&1&1&1&1&1&1&1&1&1&1&2&2&2&2&2&2&2&2&3&3&3&3&3&3&3&3\\
1&0&1&3&1&3&2&2&2&2&1&3&1&2&2&1&3&1&1&1&2&3&3&1&2&3&1&1&3\\
1&1&0&1&3&3&1&2&1&3&2&1&3&1&2&3&3&1&2&2&2&1&3&1&1&3&1&2&2\\
1&3&1&0&1&2&3&2&2&3&3&2&1&1&1&3&1&1&2&3&2&1&1&3&3&2&1&2&1\\
1&1&3&1&0&1&3&1&2&1&3&3&2&3&2&2&2&1&2&3&1&2&1&3&1&1&1&2&3\\
1&3&3&2&1&0&1&3&3&2&2&1&2&1&2&1&2&1&3&3&1&2&1&2&3&1&3&1&1\\
1&2&1&3&3&1&0&1&3&1&2&2&2&1&1&1&2&2&2&1&3&3&3&2&1&1&3&3&1\\
1&2&2&2&1&3&1&0&1&3&3&1&2&3&1&2&2&2&1&1&1&1&3&3&1&1&2&3&3\\
1&2&1&2&2&3&3&1&0&1&2&3&3&1&3&3&1&2&1&2&1&1&1&3&1&3&2&1&2\\
1&2&3&3&1&2&1&3&1&0&1&3&1&1&3&2&1&2&2&2&3&3&2&1&1&1&3&1&2\\
1&1&2&3&3&2&2&3&2&1&0&1&3&2&3&2&1&1&3&1&1&1&2&1&2&3&3&1&1\\
1&3&1&2&3&1&2&1&3&3&1&0&1&2&2&2&1&3&3&2&1&2&2&1&3&1&1&3&1\\
1&1&3&1&2&2&2&2&3&1&3&1&0&2&1&1&1&3&1&2&2&3&1&1&3&2&1&3&3\\
2&2&1&1&3&1&1&3&1&1&2&2&2&0&1&3&3&3&1&1&1&3&1&1&3&3&2&2&2\\
2&2&2&1&2&2&1&1&3&3&3&2&1&1&0&1&3&1&1&3&1&1&3&1&3&2&2&1&3\\
2&1&3&3&2&1&1&2&3&2&2&2&1&3&1&0&1&1&3&1&1&1&1&2&1&2&3&3&3\\
2&3&3&1&2&2&2&2&1&1&1&1&1&3&3&1&0&1&1&1&3&1&2&3&2&1&3&3&2\\
2&1&1&1&1&1&2&2&2&2&1&3&3&3&1&1&1&0&1&3&3&2&3&3&1&2&3&2&1\\
2&1&2&2&2&3&2&1&1&2&3&3&1&1&1&3&1&1&0&1&3&3&3&3&2&1&2&1&1\\
2&1&2&3&3&3&1&1&2&2&1&2&2&1&3&1&1&3&1&0&1&3&1&2&2&3&1&3&1\\
2&2&2&2&1&1&3&1&1&3&1&1&2&1&1&1&3&3&3&1&0&2&2&2&3&3&1&1&3\\
3&3&1&1&2&2&3&1&1&3&1&2&3&3&1&1&1&2&3&3&2&0&1&1&2&1&2&1&2\\
3&3&3&1&1&1&3&3&1&2&2&2&1&1&3&1&2&3&3&1&2&1&0&1&1&2&2&2&1\\
3&1&1&3&3&2&2&3&3&1&1&1&1&1&1&2&3&3&3&2&2&1&1&0&1&1&2&2&2\\
3&2&1&3&1&3&1&1&1&1&2&3&3&3&3&1&2&1&2&2&3&2&1&1&0&2&1&2&1\\
3&3&3&2&1&1&1&1&3&1&3&1&2&3&2&2&1&2&1&3&3&1&2&1&2&0&1&1&2\\
3&1&1&1&1&3&3&2&2&3&3&1&1&2&2&3&3&3&2&1&1&2&2&2&1&1&0&1&1\\
3&1&2&2&2&1&3&3&1&1&1&3&3&2&1&3&3&2&1&3&1&1&2&2&2&1&1&0&1\\
3&3&2&1&3&1&1&3&2&2&1&1&3&2&3&3&2&1&1&1&3&2&1&2&1&2&1&1&0
\end{bmatrix}$$

- Our software finds a R(4,3,3;29) coloring isomorphic to this one

Conclusion

- Abstraction and Symmetry are essential ingredient when modeling hard combinatorial problems
- $|R(3,3,3;13)| = 78{,}892$
- $R(4,3,3) = 30$ (two more instances remain…)