# Comparison of Multivariate Control Charts
# - A Hybrid Approach
# for Process Monitoring and Diagnosis

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# Outline

- Multivariate Process Control
- Statistical Hypothesis Test and Multivariate Control Charts
- A Hybrid Approach for Process Monitoring and Diagnosis
- Performance Comparison
- Concluding Remarks

## Multivariate Process Control

- Hotelling's $T^2$ chart is one of the most popular control charts for monitoring the mean of a multivariate normal distribution

$$T^2 \quad = \quad (\boldsymbol{x} - \boldsymbol{\mu}_0)' \, \boldsymbol{\Sigma}^{-1} \, (\boldsymbol{x} - \boldsymbol{\mu}_0) > \chi^2_{p,\alpha}$$

- Signal Diagnosis - $T^2$ Decomposition
  - Principle Component Analysis
    - Jackson, 1985
  - Bottom-up Decomposition
    - Mason, Tracy, and Young, 1992, 1995

## Multivariate Process Control

- Two alternative approaches
  - Regression-Adjusted Control Chart (Hawkins, 1991, 1993)

    $$||\boldsymbol{z}||_\infty > H$$

    where

    $$\boldsymbol{z} = diag(\sigma^{11}, \cdots, \sigma^{pp})^{-\frac{1}{2}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}_0)$$

  - Simultanuous Confidence Interval (Hayter and Tsui, 1994)

    $$||\boldsymbol{w}||_\infty > C$$

    where

    $$\boldsymbol{w} = diag(\sigma_{11}, \cdots, \sigma_{pp})^{-\frac{1}{2}}(\boldsymbol{x} - \boldsymbol{\mu}_0)$$

# Statistical Hypothesis Test
# vs.
# Multivariate Process Control

Testing Multivariate Mean Vector

$$H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0 \quad \text{against} \quad \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$$

- When shift direction is known, i.e., $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_0 + \delta \boldsymbol{d}$, the likelihood ratio (LR) statistic

  $$U(\boldsymbol{x}) = \frac{L(\boldsymbol{x}, \boldsymbol{\mu}_1)}{L(\boldsymbol{x}, \boldsymbol{\mu}_0)} \sim \boldsymbol{d}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}_0)$$

  provides a uniformly most powerful unbiased test.

  Its average run length (ARL) is

  $$ARL_{min} = \frac{1}{1 - \Phi(K - \delta) + \Phi(-K - \delta)}$$

## Global Tests - Two Principles

- Global Likelihood Ratio Test (GLRT) principle

  $$H_0: \ \boldsymbol{\mu} = \boldsymbol{\mu}_0 \quad \text{against} \quad H_1: \ \boldsymbol{\mu} \in \bigcup S^1_{\boldsymbol{d}},$$
  where $S^1_{\boldsymbol{d}} = \{\boldsymbol{\mu} : \ \boldsymbol{\mu} = \boldsymbol{\mu}_0 + \delta \boldsymbol{d}\}$.

  $$\Lambda(\boldsymbol{x}) = \frac{\sup_{\boldsymbol{\mu}} L(\boldsymbol{x}, \boldsymbol{\mu})}{L(\boldsymbol{x}, \boldsymbol{\mu}_0)}$$

- Union-Intersection Test (UIT) principle

  $$H_0: \ \boldsymbol{\mu} \in \bigcap S^0_{\boldsymbol{d}} \quad \text{against} \quad H_1: \ \boldsymbol{\mu} = \boldsymbol{\mu}_1,$$
  where $S^0_{\boldsymbol{d}} = \{\boldsymbol{\mu} : \ \boldsymbol{d}'(\boldsymbol{\mu} - \boldsymbol{\mu}_0) = 0\}$.

  For each hypotheses, a rejection region is

  $$R_{\boldsymbol{d}} = \{\boldsymbol{x} : |\boldsymbol{d}'(\boldsymbol{x} - \boldsymbol{\mu}_0)| > c_d \sqrt{\boldsymbol{d}' \boldsymbol{\Sigma}^{-1} \boldsymbol{d}}\}.$$

  The union of all rejection regions is

  $$R = \bigcup R_{\boldsymbol{d}}$$

## Global Tests

Both principles lead to the same monitoring statistic

$$\lambda(\boldsymbol{x}) = (\boldsymbol{x} - \boldsymbol{\mu}_0)' \, \boldsymbol{\Sigma}^{-1} \, (\boldsymbol{x} - \boldsymbol{\mu}_0)$$

- Statistical sense: mean vector changes or not
- Variable diagnosis
- Likelihood of which variable(s) experiences changes

## Variable Screening

- GLRT Principle

$$H_0: \ \boldsymbol{\mu} = \boldsymbol{\mu}_0 \quad \text{against} \quad H_1: \ \boldsymbol{\mu} \in \bigcup_{\wp \in \mathcal{P}} S^1_{\wp},$$

where $S^1_{\wp} = \{\boldsymbol{\mu} : \boldsymbol{\mu}_{\wp} \neq \mathbf{0}_{\wp} \text{ and } \boldsymbol{\mu}_{(\wp)} = \mathbf{0}_{(\wp)}\}$.

For each individual hypothesis,

$$\tilde{\lambda}_{\wp}(\boldsymbol{x}) = \lambda(\boldsymbol{x}) - \lambda_{(\wp)}(\boldsymbol{x}) \sim \chi^2_{|\wp|}$$

$U^2$ statistic - Runger, 1996

**This principle reduces to the global $T^2$ statistic.**

## Variable Screening

- UIT Principle

$$H_0: \ \boldsymbol{\mu} = \bigcap_{\wp \in \mathcal{P}} S_\wp^0 \quad \text{against} \quad H_1: \ \boldsymbol{\mu} \neq \mathbf{0},$$

where $S_\wp^0 = \{\boldsymbol{\mu} | \boldsymbol{\mu}_\wp = \mathbf{0}_\wp\}$,

For each individual hypotheses,

$$\lambda_\wp = \boldsymbol{x}_\wp \boldsymbol{\Sigma}_{\wp\wp}^{-1} \boldsymbol{x}_\wp$$

The UIT rejection region is

$$R_\wp = \{\boldsymbol{x} : \bar{\lambda} = \lambda_\wp(\boldsymbol{x}) / \chi^2_{|\wp|,\alpha} > 1\}$$

**However,**

$$P_0\{\bigcup R_\wp\} > \alpha$$

# A Hybrid Approach

- Some Observations:
  - No principle dominates the other in detecting all directions of shifts
  - GLRT reduces the variation of a set of variables by using its correlation information with other variables while maintains the signal magnitude
  - GLRT allocates unequal Type I errors to all hypothesis components

    Small type I error $\quad \rightarrow \quad$ small group

- Proposal: Assign type I errors to all testing components as equally as possible

  Assuming a priori likelihood/probability $q_{\wp}^{0}$ of potential changes for each variable subset $\wp$,

$$\lambda_{\wp}^{HB}(\boldsymbol{x}) = \frac{q_{\wp}^{0}\tilde{\lambda}_{\wp}(\boldsymbol{x})}{\chi^2_{|\wp|,\alpha}} > c.$$

## A Simplified Procedure

- Using a threshold to screen out un-change variables

$$\boldsymbol{J}^{-} = \{j : |x_j/\sigma_{jj}| \le C\}$$

- Segmenting $\boldsymbol{P}$ into two subsets of variables

$$\boldsymbol{P} = \boldsymbol{J} \bigcup \boldsymbol{J}^{-}$$

- If $\boldsymbol{J} = \boldsymbol{P} \backslash \boldsymbol{J}^{-} = \emptyset$, all variables are in-control.

- If $\boldsymbol{J} \neq \emptyset$, monitoring the standardized $T^2$ statistic of the regression-adjusted variables of $\boldsymbol{J}^{-}$,

$$\lambda^{ADP}(\boldsymbol{x}) = \frac{\tilde{\lambda}_{\boldsymbol{J}}(\boldsymbol{x})}{\chi^2_{|\boldsymbol{J}|,\alpha}} > H.$$

## Properties of the Simplified Approach

- When $C = 0$, it is exactly the $T^2$ chart.

- When $C \geq C_{HT}$, it is Hayter and Tsui's M chart.

- When $0 < C < C_{HT}$, it is something similar to the hybrid chart.

## Performance Comparison

- Benchmark: UMP test given shift direction

  Relative Efficiency:

$$RE = \frac{ARL}{ARL_{min}}$$

- Mean Shift Magnitude:
  - Quality Measure

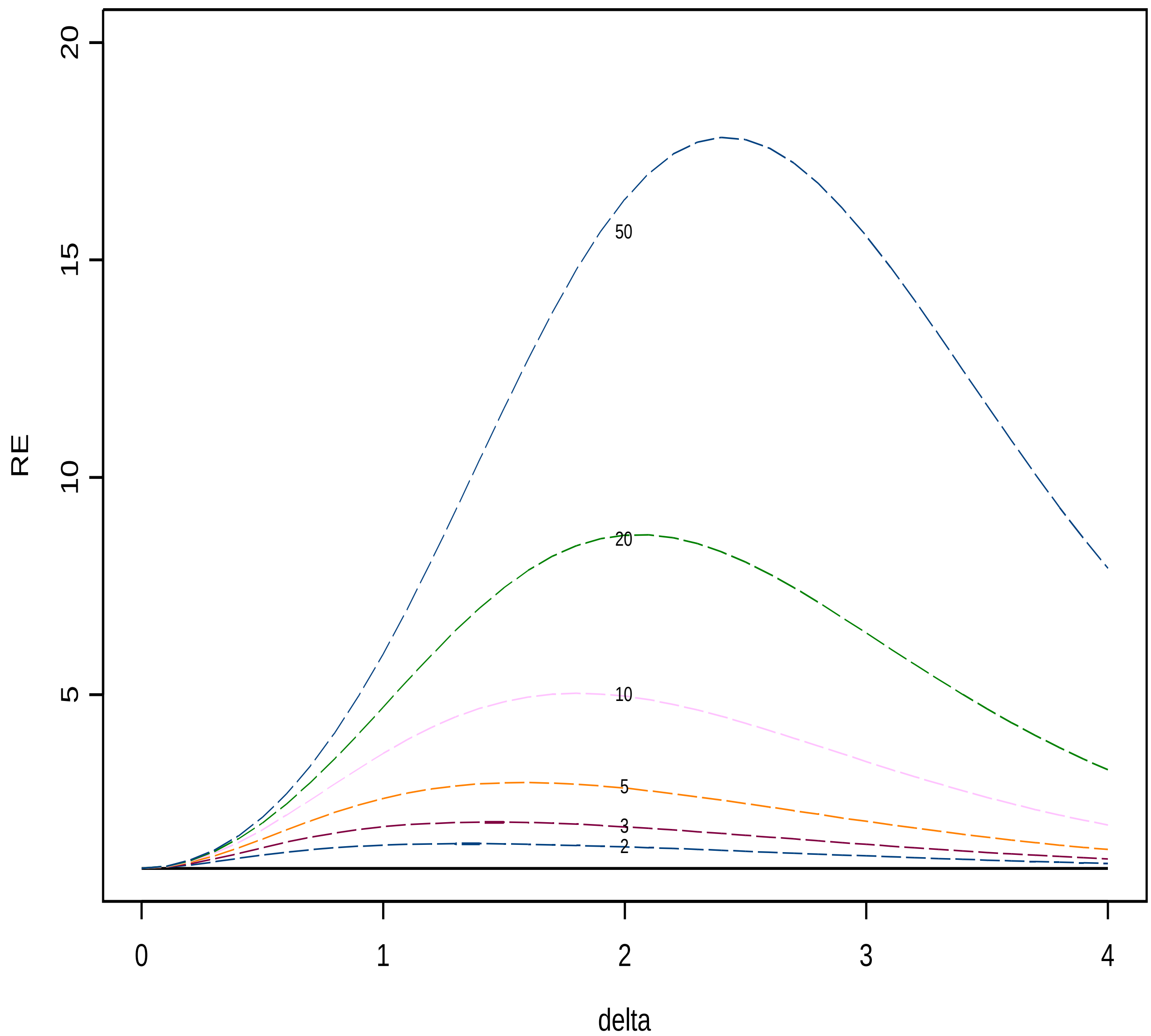
$$(\boldsymbol{x} - \boldsymbol{\mu}_0)' \ (\boldsymbol{x} - \boldsymbol{\mu}_0)$$

  - Statistical Measure

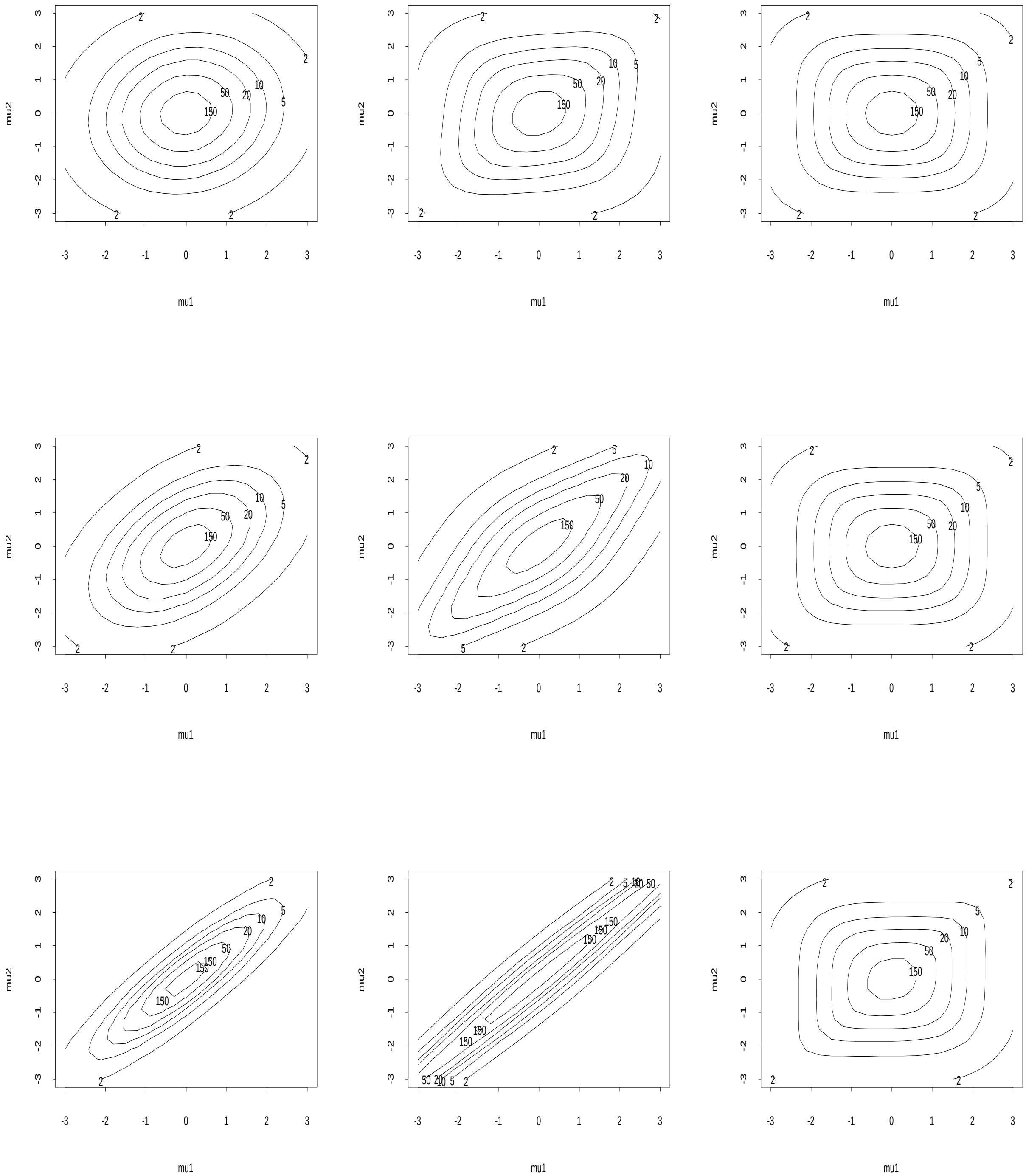
$$(\boldsymbol{x} - \boldsymbol{\mu}_0)' \, \boldsymbol{\Sigma}^{-1} \, (\boldsymbol{x} - \boldsymbol{\mu}_0)$$

# RE - Curse of Dimensionity

## $T^2$ chart

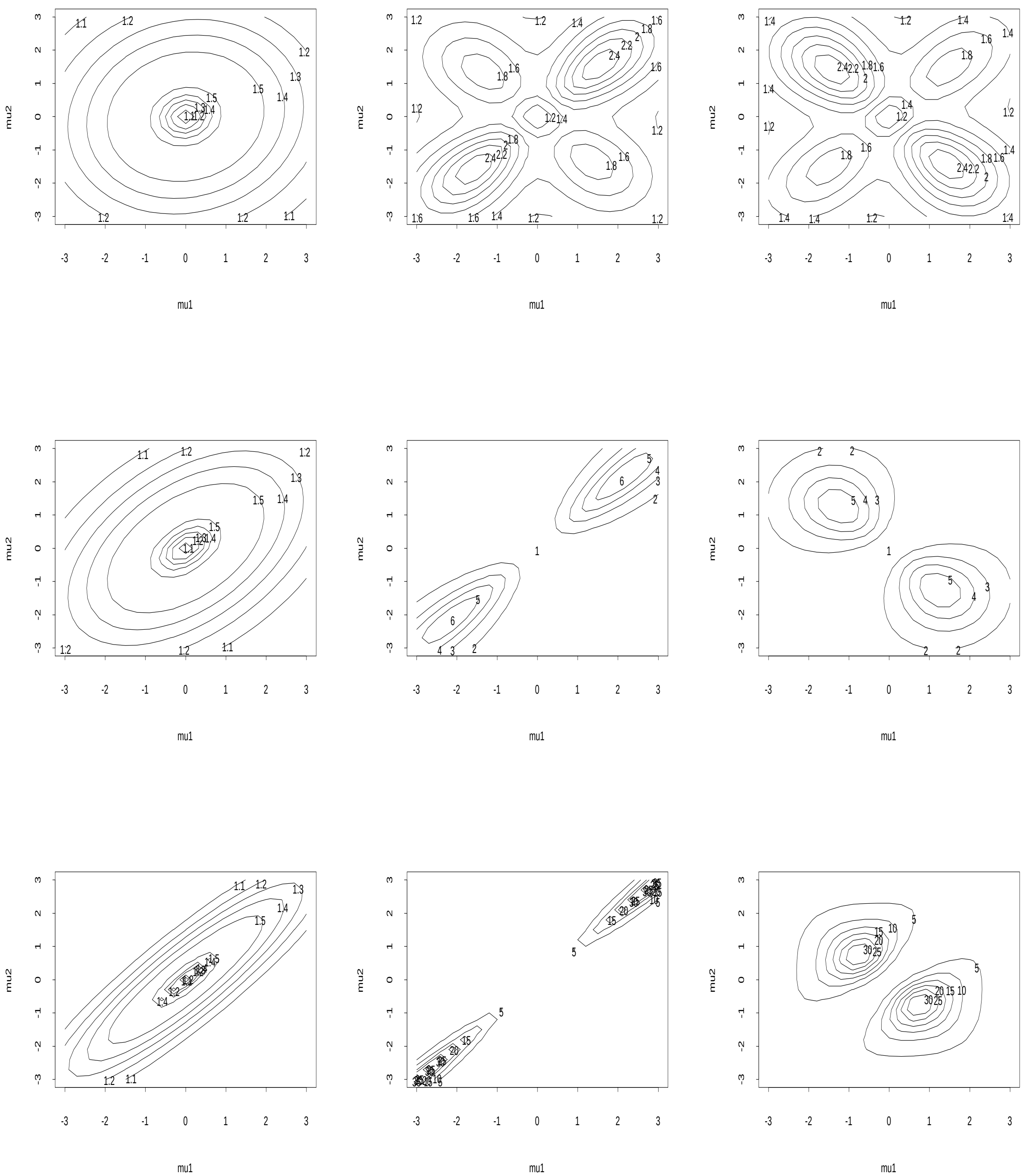


# ARL - Quality View
## $T^2$, Regression-Adjusted, M charts



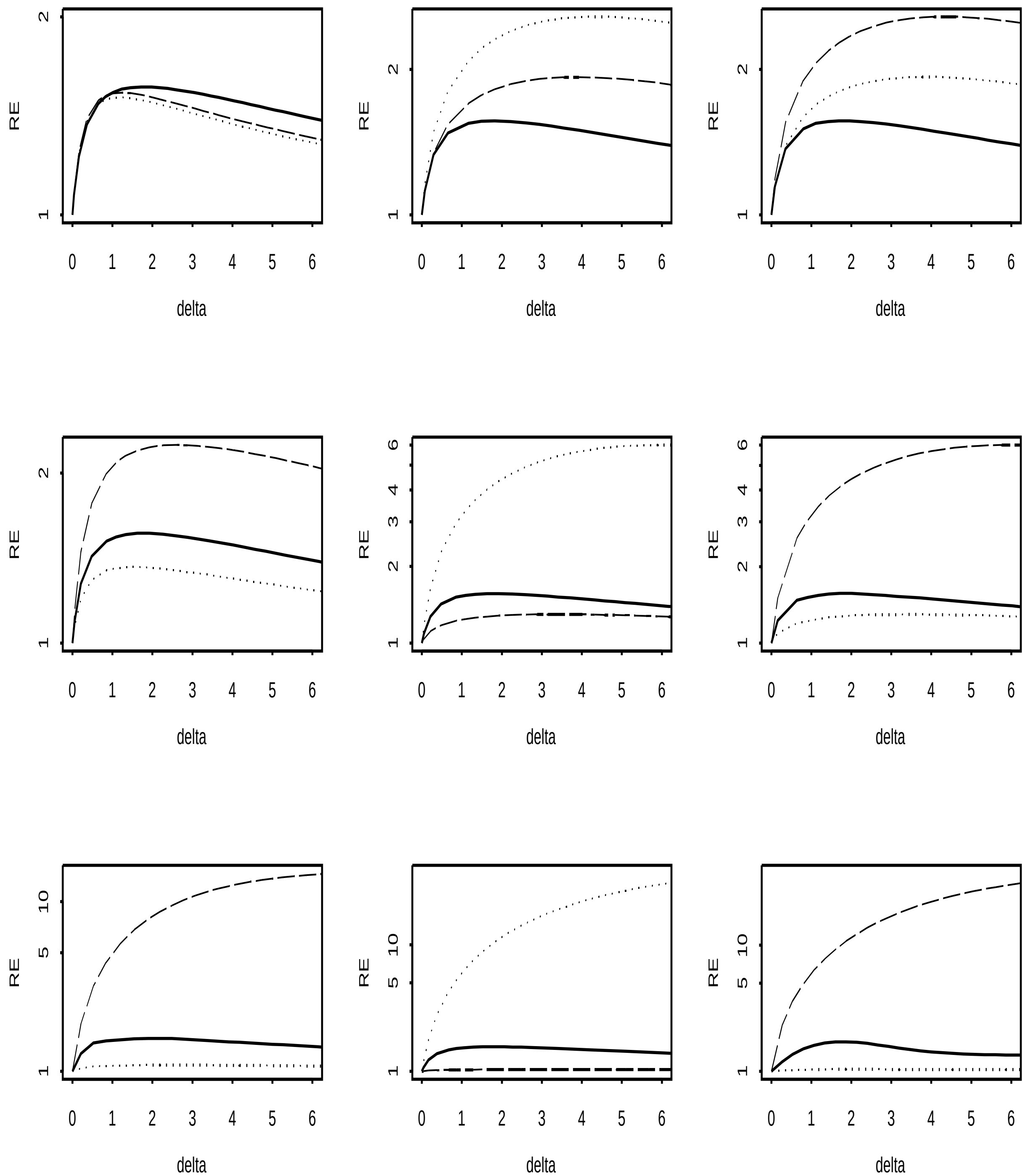

(a) $T^2$ chart (b) Regression-Adjusted chart (c) M chart

# RE - Contour Plots of Statistical View
## $T^2$, Regression-Adjusted, M charts



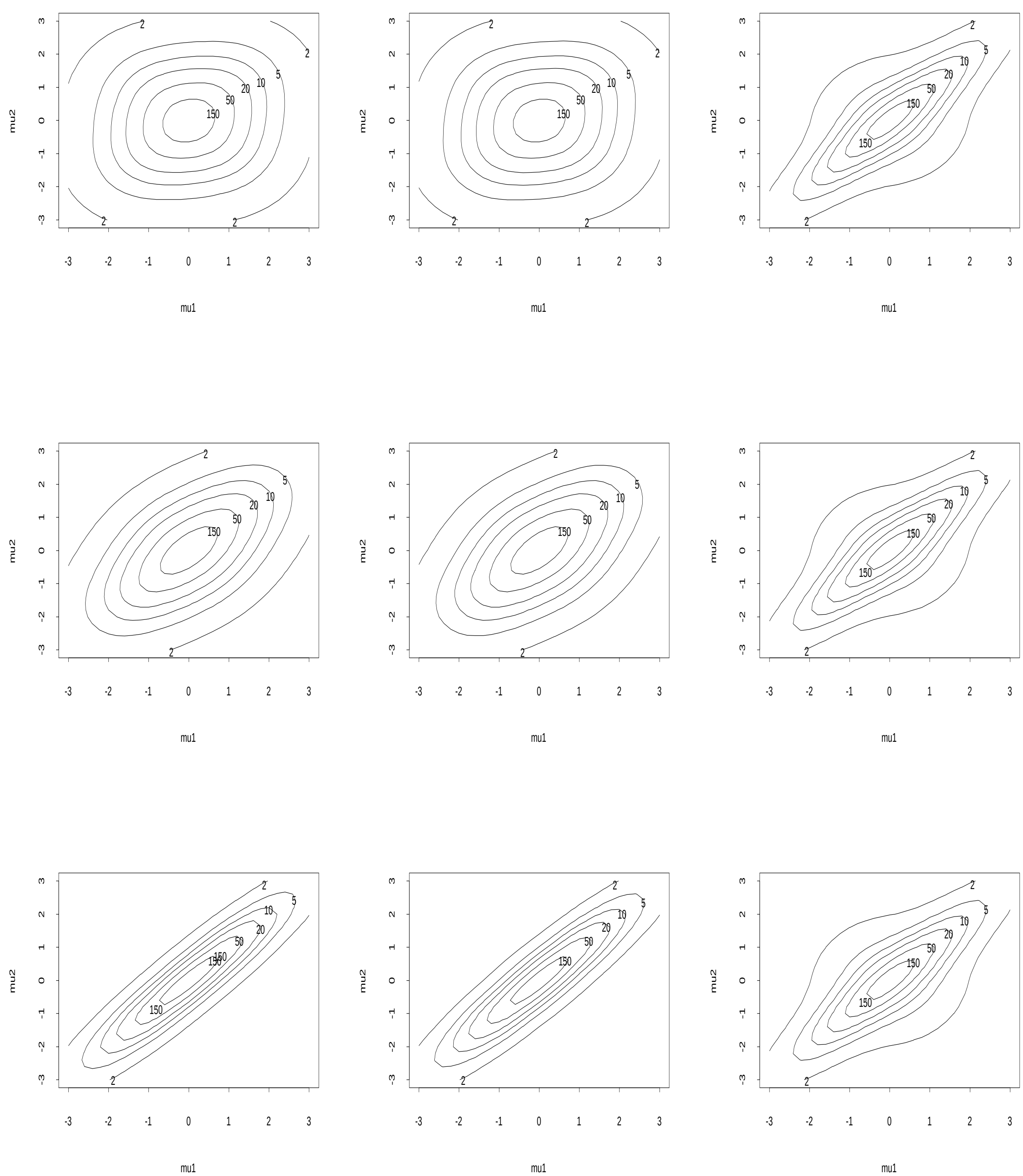

(a) $T^2$ chart (b) Regression-Adjusted chart (c) M chart

# RE - Some Special Cases
## $T^2$, Regression-Adjusted, M charts




(a) $\mu_1 = 0$ (b) $\mu_1 = \mu_2$ (c) $\mu_1 = -\mu_2$

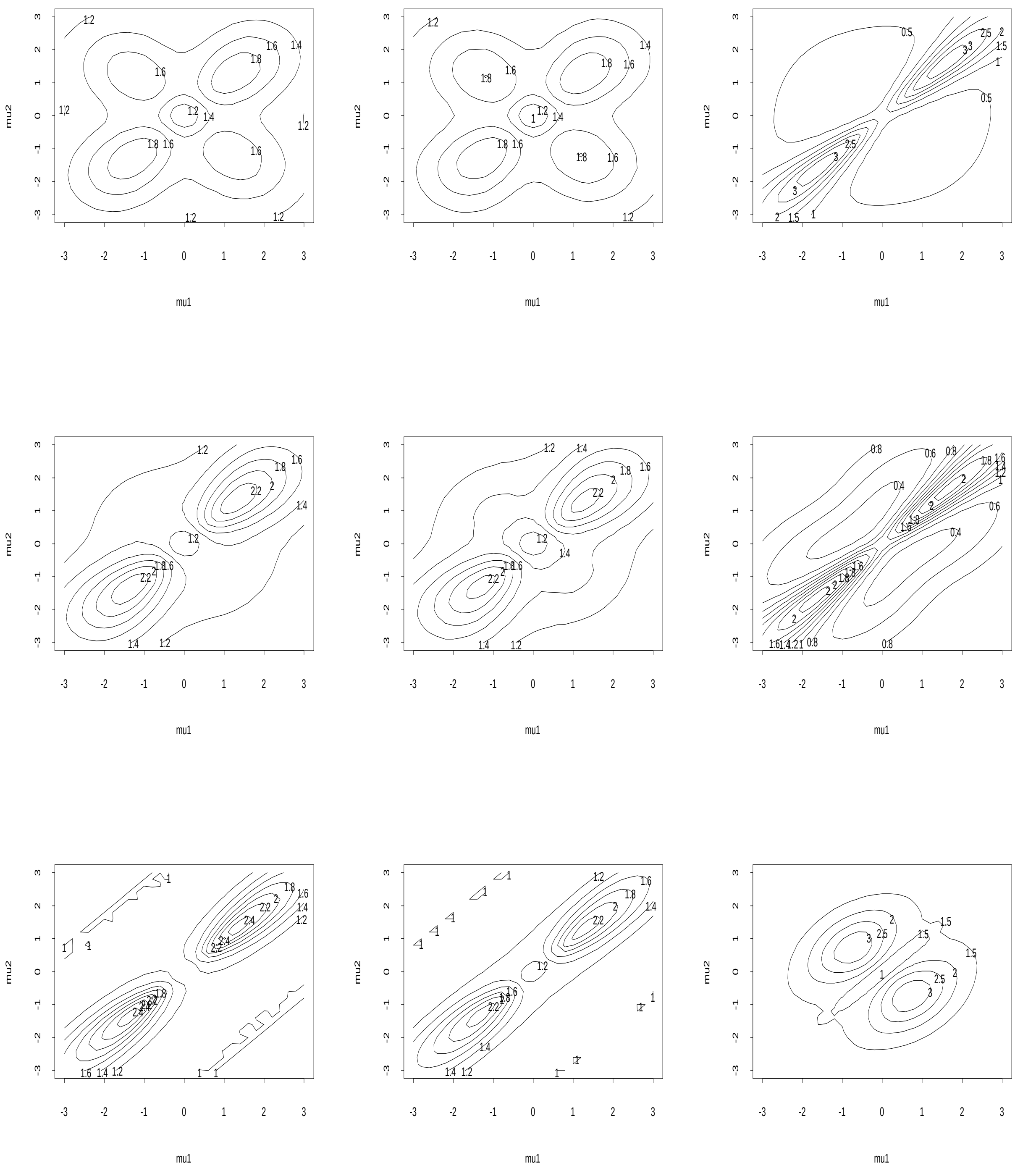
# ARL - Quality View
## Hybrid charts




(a) Hybrid chart (b) Adaptive chart $C = 1$ (c) Adaptive chart $C = 2$

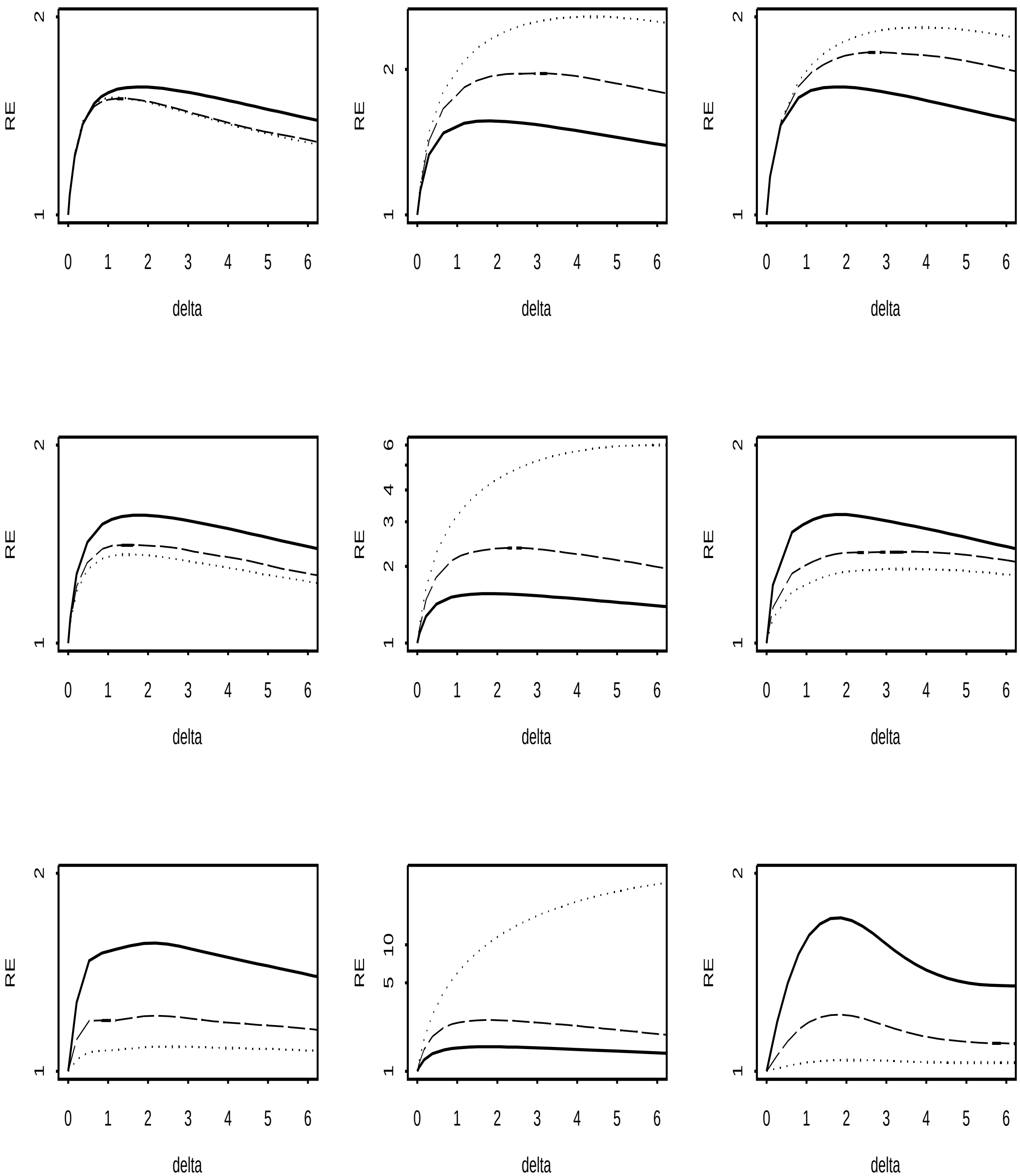
# RE - Contour Plots of Statistical View

## Hybrid charts



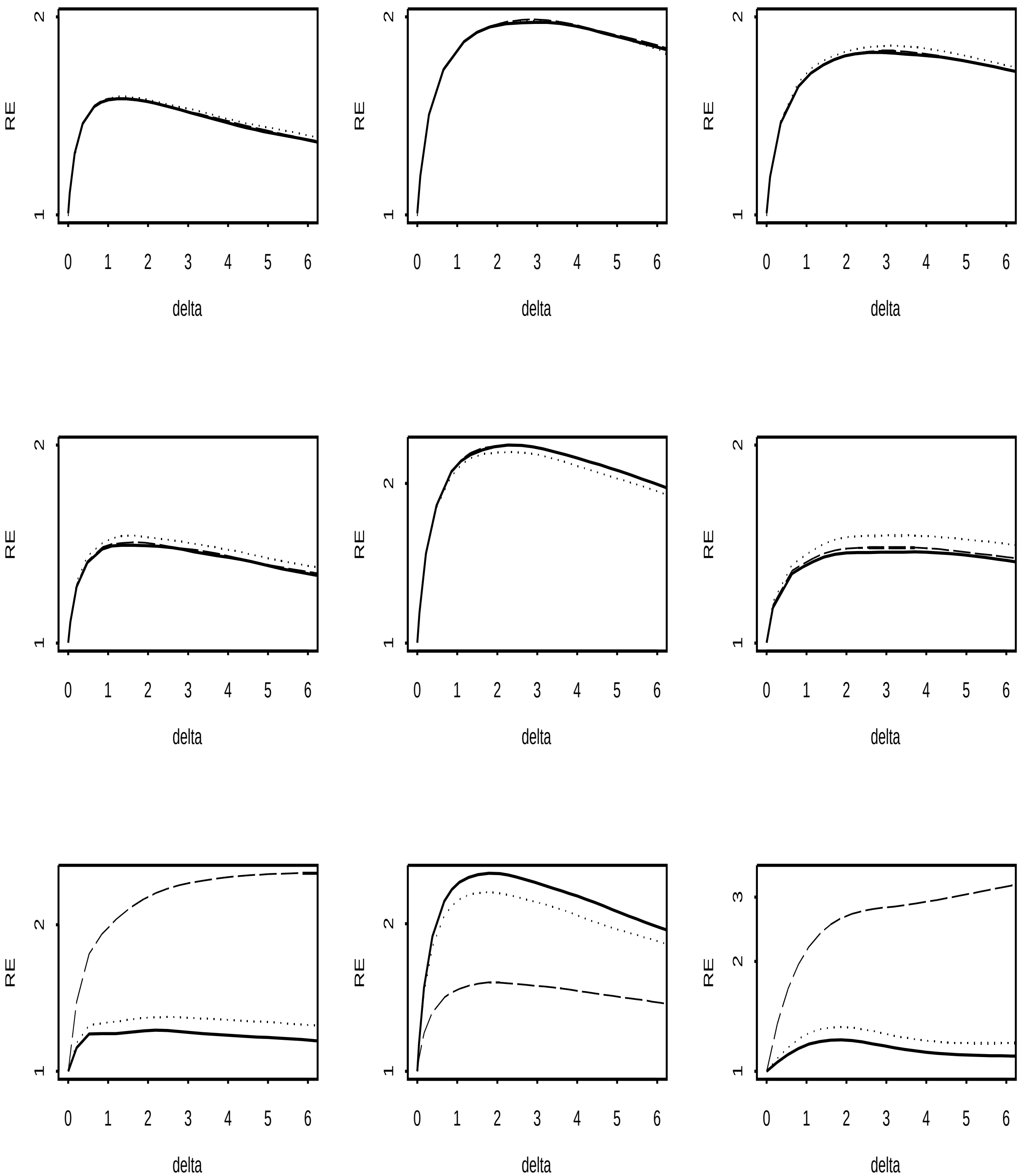

(a) Hybrid chart (b) Adaptive chart $C = 1$ (c) Adaptive chart $C = 2$

# RE - Some Special Cases
## $T^2$, Regression-Adjusted, Hybrid charts




(a) $\mu_1 = 0$ (b) $\mu_1 = \mu_2$ (c) $\mu_1 = -\mu_2$

# RE - Some Special Cases
## Hybrid charts and Simplification




(a) $\mu_1 = 0$ (b) $\mu_1 = \mu_2$ (c) $\mu_1 = -\mu_2$

## Conclusions and Future Research

- A hybrid method is proposed based on the GLRT and UIT principles to integrate multivariate process monitoring and diagnosis.

- An adaptive approach is introduced to simplify the hybrid method in practice.

- The diagnosis of signals from the adaptive approach is simpler than the $T^2$ decomposition. Needs further investigations.