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Sampling Schemes

Sampling Errors in Some Global Climate
Sampling by Polar-Orbital Satellites • Sampling by Point-Gauge Networks • Measures of Sampling Error • Linear Estimation and Aliasing Effects • Spherical Harmonic Representation • Overview
Estimation and change detection
Data archiving and data compression
Spectral methods in some GCMs' numerical schemes

Applications:

\[ n = m \text{ zonal wavenumber} \]
\[ \ell = \text{ degree} \]
\[ (u)^{\omega m}_\lambda (\ell)^{\omega m}_\ell \prod_{j=0}^{j=\ell} \sum_{m=0}^{m=\ell} (\ell, m, n) \]

Laplace Series Expansion: For an anomaly field \( \mathcal{T}(u, \ell) \),

Spheriral Harmonic Representation
\[(m \cdot \ell\)_{m} \sum_{\ell} = w_{\ell}\]

- Linear Estimation: weighted average of observations

\[(f, \cdots, f = f)\]

- Point-Gauge Observations: simultaneous measurements

\[(u)_{\nu \ell} (u)_{m}^{w_{\ell}} (f, u)_{f} \int = (f)_{w_{\ell}} \]

- Spherical Harmonic Coefficients

**Estimation of S. H. Coefficients**
\[
\begin{aligned}
&\sum_{\ell} (\ell u)_{\mu}^* \Lambda (\ell u)_{\mu} \partial_{\lambda} \sum_{\mu} \frac{1}{\ell} = (\ell m, \ell, \ell) L \\
&\sum_{\ell} (\ell m, \ell, \ell) \sum_{\ell} \sum_{\ell} \sum_{\ell} = \sum_{\ell} \sum_{\ell} \sum_{\ell}
\end{aligned}
\]
\[ W / 2^\nu \times \text{Gaussian weights} = \ell m \]
\[ (\phi^N)_{(\sin)} = b \phi \]

---

Gaussian-Legendre Network:

\[ (N, I = b) \quad (I + N) / b \nu + \frac{2}{\nu -} = b \phi \]

Latitute:

\[ (N, I = d) \quad N / d^2 \nu + \nu - = d \phi \]

Longitude:

\[ N W = f \]

---

Gaussian-Legendre versus Uniform
(unbiased) \[ N > \gamma \geq |m| \quad \text{if} \quad \gamma \in \mathbb{N} \]

\[ 0 \neq \gamma - N > |m| \implies 2r \gamma - 0 = 0 \quad \text{if} \quad \gamma \neq 2r \quad \text{or} \quad m \neq 0 \]

Gauss-Legendre Case: with \[ \mathcal{W} = N \]

General Case: unit, longitude, symm, latitude, even \[ \mathcal{W} \]

STRUCTURE OF ALIASING COEFFICIENTS
\[ e_{lm}^2 = \frac{E\{ \hat{T}_{lm}^2 \}}{\text{Var}\{T_{lm}\}} \]

\( e_{lm}^2 \) as a fraction of the "size" of \( T_{lm} \)

**Mean-Squared Error:** a statistical index of error

**MEASURES OF SAMPLING ERROR**

- Relative MSE: MSE as a fraction of the "size" of \( T_{lm} \)
depends only on the field's statistics

$\mu_{\mathcal{L}}$ power of the aliases

$$\{ \mu_{\mathcal{L}} \} \text{var} = \frac{\mu_{\mathcal{L}}}{2}$$

depends only on the sampling design

$\text{sqared aliasing function (SAF)}$

$$|^{m_i - \mu_{\mathcal{L}} - \varphi \mu - \varphi} \mathfrak{A}(m_i, m_j, r_i, r_j) \mathfrak{F} = \mathfrak{F}$$

Factorization Property

$$\frac{\varphi}{2} \mathfrak{A}(m_i, m_j, r_i, r_j) \mathfrak{F} \sum_{\mathfrak{F}} \sum_{\mathfrak{F}} = \mu_{\mathcal{L}}$$

Aliased Power (assuming homogeneity)

Composition of Sampling Error
Interpretation: The fraction of MSE that can be attributed to the $2^\vartheta + 1$ spherical harmonic components of degree $\vartheta$ is

\begin{align*}
I &= (\mu)^{\omega \vartheta p} \underbrace{\sum_{\infty}^{0=\vartheta}}_{(\text{dimensionless})} \\
\frac{\partial^\vartheta}{\partial \omega^\vartheta} (\vartheta, 'm, \vartheta, \vartheta, \vartheta) \underbrace{\sum_{\vartheta}^{\vartheta-\infty} \vartheta \omega \vartheta}_{\text{MSE Spectrum: distribution of the aliased power}}\frac{2}{I} &= (\mu)^{\omega \vartheta p}
\end{align*}

Composition of MSE (cont'd)
\[
\frac{z \{(I + g)^0r^l + I\}}{\frac{\vartheta l}{4n o^l}} = \vartheta^o
\]

Statistical Parameters:

\[
d_l \vartheta^o = \{(u)_F\} \text{ var, zero-mean white noise, spatial scale parameter}
\]

(u)_F = (u)_L + (u)_L \Delta^0r_l -

Energy Balance Model (EBM):

A simple climate model
Relative MSE of 98-Station Networks ($N = \frac{2W}{l}$)

<table>
<thead>
<tr>
<th>l = 1</th>
<th>l = 2</th>
<th>l = 3</th>
<th>l = 4</th>
<th>l = 5</th>
<th>l = 6</th>
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<tr>
<td>0.08</td>
<td>0.82</td>
<td>0.09</td>
<td>0.42</td>
<td>0.30</td>
<td>0.30</td>
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<tr>
<td>1.02</td>
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<td>0.92</td>
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<td>0.07</td>
<td>0.45</td>
<td>0.91</td>
<td>0.39</td>
<td>0.29</td>
<td>0.30</td>
</tr>
<tr>
<td>1.05</td>
<td>1.42</td>
<td>0.97</td>
<td>0.45</td>
<td>0.31</td>
<td>0.30</td>
</tr>
<tr>
<td>0.29</td>
<td>0.59</td>
<td>1.04</td>
<td>0.48</td>
<td>0.32</td>
<td>0.30</td>
</tr>
<tr>
<td>0.09</td>
<td>0.51</td>
<td>1.09</td>
<td>0.54</td>
<td>0.33</td>
<td>0.30</td>
</tr>
</tbody>
</table>

Lat.-Long. Uniform Network

Relative MSE for Estimating $T^{0.0}$
non-simultaneous (global) sampling in time
— non-complete (short-time) coverage in space

Sampling Properties:

\[ T(n) = \mathcal{I} \in t \cap \mathcal{I} \]

Satellite Observations: geo-asynchronous satellites

SAMPLING BY SATELITE
\[ \mathcal{W}(\mathcal{I}) \int |\mathcal{I}| \cdot \frac{\mathcal{I}}{I} = \mathcal{W}_\mathcal{I} \]

- time-averaged coefficients

\[ \mathcal{I} \in O \] for some \( t_0 \) ∈ \( \mathcal{I} \)

- instantaneous coefficients \( \mathcal{W}(\mathcal{I}) \) for some \( t_0 \) ∈ \( \mathcal{I} \)

Estimation:

\[ \mathcal{W}(\mathcal{I}) \int \left( (\mathcal{I}) \mathcal{U} \right) \mathcal{W}_\mathcal{I} \int \mathcal{I} = \mathcal{W}_\mathcal{I} \]

Linear Estimation:

ESTIMATION USING SATELLITE DATA
\[(\mathcal{M}, \vartheta)_{f}^{(m', \varrho)_A}(m', \varrho, m', \varrho)_{A(SAF) \text{ Factorization Property}} \]

(assuming homogeneity and stationarity)

\[
m_{D} (m', \varrho)_{f} (m', \varrho, m', \varrho)_{A} \int_{-\infty}^{\infty} \sum_{\varrho} \sum_{\infty} = \frac{m_{2}}{2}
\]

Mean-Squared Error: Composed of aliased power

SAVING ERRORS AND ALIASING EFFECTS
\[
(m', \vartheta)_{\text{temporal MSE spectrum}} \triangleq \lim_{\vartheta \to 0} \sum_{\vartheta} (m)_{\text{temporal MSE spectrum}}
\]

Spherical MSE spectrum: distribution over degrees

\[
\varrho \left( (m', \vartheta)_{\text{spherical MSE spectrum}} \right) \int_{\varrho} = (\vartheta)_{\text{spherical MSE spectrum}}
\]

Joint MSE spectrum: distribution over frequencies

\[
(m', \vartheta) \triangleq \sum_{\vartheta} \frac{\varrho(m', \vartheta)}{\text{I}} = (m', \vartheta)_{\text{joint MSE spectrum}}
\]
without repetition
the Earth times in \( N \) days to complete its orbital pattern

Remark: the polar-orbiting satellite is assumed to encircle

\[
\frac{\omega}{(\varpi \cos \varpi \cos \varphi)} = (\varpi \theta)
\]

\[
\omega = (\varpi \theta)
\]

\[
\sin(\sin(\varphi)) = (\varpi \phi)
\]

Sampling Design: with \( a = 2\pi \frac{N}{M} \),

Polar-orbiting Satellites
\[
\frac{\text{#}((1 + \gamma) \partial_t^0 \hat{\mathbf{x}} + \mathbf{I}) + \xi t^0 \partial_t^0 \hat{\mathbf{x}}}{\mu / \nu} = (m)^g
\]

**Power Spectral Density:**

- Spatial scale parameter: \(0\)
- Temporal scale parameter: \(0\)

\[
\frac{\xi}{\xi_o} = \{(t, u) \}
\]

White noise, \text{Var}:

\[
(t, u) = (t, u) \mathcal{I} + (t, u) \mathcal{L} \Delta^0 \mathcal{I} - (t, u) \mathcal{L}^0 \mathcal{I}
\]

**Time-Dependent EBM:**

**A Simple Climate Model**
- Relative MSE decreases for long-term averages only if the time scale of the field is sufficiently large.

<table>
<thead>
<tr>
<th>Relative MSE for Estimating $\tilde{\theta}_0$ of EBM Fields</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average period (days)</td>
</tr>
<tr>
<td>Time scale $\tau$ (days)</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>10</td>
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<tr>
<td>20</td>
</tr>
<tr>
<td>30</td>
</tr>
<tr>
<td>40</td>
</tr>
</tbody>
</table>
The lack of zonal resolution is largely responsible for the splitting of waves $m(\hat{m} = 0)$ than with the number of meridional waves $m(\hat{m} = 0)$. MSE increases faster with the number of zonal waves $m(\hat{m} = 0)$. MSE increases as the wavenumber increases. MSE decreases as the time scale $T_0$ increases.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$T_0$ = 30 days</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.350</td>
<td>2.612</td>
</tr>
<tr>
<td>0.259</td>
<td>0.967</td>
</tr>
<tr>
<td>0.346</td>
<td>0.059</td>
</tr>
<tr>
<td>0.136</td>
<td>0.122</td>
</tr>
<tr>
<td>0.179</td>
<td>0.071</td>
</tr>
<tr>
<td>0.069</td>
<td>0.124</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$m$</th>
<th>$T_0$ = 10 days</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.313</td>
<td>0.124</td>
</tr>
<tr>
<td>0.310</td>
<td>0.071</td>
</tr>
<tr>
<td>0.089</td>
<td>0.122</td>
</tr>
<tr>
<td>0.111</td>
<td>0.071</td>
</tr>
<tr>
<td>0.069</td>
<td>0.124</td>
</tr>
</tbody>
</table>

Relative MSE for estimating $\hat{m}$ of EBM fields ($m = N = 16, N = 1$).
Future Research: (a) analyzing observations from multiple signal variations on sampling errors

• For point-gauge networks and asynchronous satellites

Methodology: a unified method to analyze sampling errors

CONCLUSIONS