

A New Approximation Method for Set Covering Problems, with Applications to Multidimensional Bin Packing*

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Abstract

In this paper we introduce a new general approximation method for set covering problems, based on the combination of randomized rounding of the (near-)optimal solution of the *Linear Programming* (LP) relaxation, leading to a partial integer solution, and the application of a well-behaved approximation algorithm to complete this solution. If the value of the solution returned by the latter can be bounded in a suitable way, as is the case for the most relevant generalizations of bin packing, the method leads to improved approximation guarantees, along with a proof of tighter integrality gaps for the LP relaxation.

For d -dimensional vector packing, we obtain a polynomial-time randomized algorithm with asymptotic approximation guarantee arbitrarily close to $\ln d + 1$. For $d = 2$, this value is $1.693\dots$, i.e., we break the natural 2 “barrier” for this case. Moreover, for small values of d this is a notable improvement over the previously-known $O(\ln d)$ guarantee by Chekuri and Khanna [8].

For 2-dimensional bin packing with and without rotations, we obtain polynomial-time randomized algorithms with asymptotic approximation guarantee $1.525\dots$, improving upon previous algorithms with asymptotic performance guarantees arbitrarily close to 2 by Jansen and van Stee [17] for the problem with rotations and $1.691\dots$ by Caprara [5] for the problem without rotations.

The previously-unknown key property used in our proofs follows from a retrospective analysis of the implications of the landmark bin packing approximation scheme by Fernandez de la Vega and Lueker [13]. We prove that their approximation scheme is “subset oblivious”, which leads to numerous applications.

1 Introduction

We analyze a simple method to find approximate solutions to Set Covering problems, showing that it leads to improved approximation guarantees for the two multidimensional generalizations of Bin Packing that are most relevant for practical applications and whose study goes back to the origins of Operations Research.

The first generalization is d -Dim(ensional) Vector Packing. Here each item and bin is a d -dimensional vector with non-negative entries, and the goal is to pack the items using the minimum number of bins so that for every bin the sum of the vectors packed in that bin is coordinate-wise no greater than the bin’s vector. This problem is widely used to model resource allocation problems. The items can be viewed as jobs with requirements for d independent resources such as memory, CPU, hard disk, \dots , and the bins as machines that have a certain amount of each resource available. The goal is then to place the jobs on

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the minimum number of machines so that no machine is overloaded and the requirements of each job are met.

The second generalization is 2-Dim Bin Packing, where a given set of rectangular items must be packed into the minimum number of unit rectangular bins. The most common case is the so called orthogonal packing, where the edges of the items must be packed parallel to the sides of the bin, and the items are either allowed to be rotated by 90 degrees or not. In the rest of the paper, by “rotation” we will always mean a “90-degree rotation”, also called orthogonal rotation. A closely related problem is 2-Dim Strip Packing (also known as Cutting-Stock), motivated by applications in cloth cutting and steel cutting industry. Here we are given a strip of infinite width and finite height, and the goal is to pack the items into the strip so that the width occupied is minimized.

Literature review

For 1-Dim Bin Packing, relevant results were obtained as soon as the main concepts in approximation were defined, in the late 1970s and early 1980s. These results essentially settled the status of Bin Packing with the *Asymptotic Polynomial-Time Approximation Schemes* (APTASes) due Fernandez de la Vega and Lueker [13] and Karmarkar and Karp [22].

For d -Dim Vector Packing, a folklore asymptotic approximation guarantee arbitrarily close to d follows trivially by considering for each dimension the items that have the largest component in that dimension and packing these items into bins by applying an APTAS for 1-Dim Bin Packing. The first non-trivial result was due to Chekuri and Khanna [8] who gave, for constant d , a polynomial-time algorithm with approximation guarantee $O(\ln d)$. (Though [8] state their result as $O(\ln d)$, upon a closer look, the asymptotic approximation guarantee of their method is actually about $\ln d + 2$ for large d .) On the other hand, Woeginger [28] ruled out an APTAS even for $d = 2$. For $d = 2$ the best known result is an absolute approximation guarantee of 2 due to Kellerer and Kotov [19]. A natural question motivated by its intrinsic simplicity and practical applications [7] is whether there is a polynomial-time algorithm with asymptotic approximation guarantee better than 2 for $d = 2$ (the method of [8] has guarantee 3 for $d = 2$).

Although 2-Dim Bin Packing and Strip Packing are fairly complex, starting from the 1980s a slow but continuous progress was made, which culminated in a series of recent relevant results. Kenyon and Rémila [21] showed that there is an APTAS for 2-Dim Strip Packing without rotations. This was recently extended by Jansen and van Stee [17] to the case with rotations. For 2-Dim Bin Packing, Bansal et al. [1] showed that it does not admit an APTAS unless $P=NP$. The best known polynomial-time approximation algorithm for 2-Dim Bin Packing without rotations is due to Caprara [5] and has asymptotic approximation guarantee 1.691 . . . For the case with rotations, an asymptotic approximation guarantee arbitrarily close to 2 follows from the result of [17]. An APTAS is known for the Guillotine 2-Dim Bin Packing [4], in which the items must be packed in a certain structured way.

Contribution and outline

Our general method for a Set Covering problem works as follows. First of all, the LP relaxation of the problem is solved. Then, a randomized rounding procedure is applied for a few steps, after which one is left with a “small” fraction of uncovered elements (called the residual instance). Finally, these elements are covered using some approximation algorithm.

We prove that, if the approximation algorithm used in the last step has asymptotic approximation guarantee ρ and satisfies certain properties (being *subset oblivious*, see Section 3), then the overall method is a randomized algorithm with asymptotic approximation guarantee arbitrarily close to $\ln \rho + 1$. Roughly speaking, subset oblivious means that not only the algorithm produces a solution with value

at most $\rho \text{opt}(I)$ on instance I , but also, given a “random” subset S of I where each element occurs with probability about $1/k$, the value of the solution produced by the algorithm on S is bounded by approximately $\rho \text{opt}(I)/k$.

The key observation is that many known algorithms for Bin Packing problems are subset oblivious, or can be modified to be such. This leads to the following results based on our general method, applied by formulating the problem at hand as a Set Covering problem, each set corresponding to a valid way of packing a bin and the goal being to cover all the items with the minimum number of sets.

We first show that the classic APTAS for 1-Dim Bin Packing due to Fernandez de la Vega and Lueker [13] is a subset-oblivious algorithm after minor modifications. Based on this, we give a simple subset-oblivious algorithm for d -Dim Vector Packing for constant d with asymptotic approximation guarantee arbitrarily close to d . Plugged into our general method, this leads to a polynomial-time randomized algorithm with asymptotic approximation guarantee arbitrarily close to $\ln d + 1$ for fixed d . For small values of d this is a notable improvement over the previously known $O(\ln d)$ guarantee [8] mentioned above. For $d = 2$, our result implies an asymptotic approximation guarantee of $\ln 2 + 1 = 1.693\dots$ which breaks the natural barrier of 2 for this case.

For 2-Dim Bin Packing with and without rotations, we give subset-oblivious algorithms with asymptotic approximation guarantee $1.691\dots$. Note that, in itself, this is an improvement for the case with rotations. Plugged into our general method, these lead to a polynomial-time randomized algorithm with asymptotic approximation guarantee arbitrarily close to $\ln(1.691\dots) + 1 = 1.525\dots$, improving on the above mentioned $1.691\dots$ for the case without rotations [5] and (arbitrarily close to) 2 for the case with rotations [18].

In the description above, we assumed that an optimum solution of the Set Covering LP relaxation was available. However, since our sets are implicitly described and are typically exponentially many, the problem of solving this LP relaxation is non-trivial. For the applications considered in this paper, we show that the LP relaxation can be solved to within $(1 + \varepsilon)$ accuracy for any $\varepsilon > 0$. For d -Dim Vector Packing, we do this by observing that the dual separation problem (also known as column generation problem) has a *Polynomial-Time Approximation Scheme* (PTAS), which implies a PTAS for the LP relaxation following the framework of [22, 26, 15, 16]. However, this approach does not work for 2-Dim Bin Packing. In this case the dual separation problem is the well-known maximum 2-Dim (Geometric) Knapsack problem, for which the best known algorithm, due to Jansen and Zhang [18], has a performance guarantee arbitrarily close to 2, and the existence of a PTAS is open. However, in a companion paper [2] we illustrate a PTAS for a suitable restriction of 2-Dim Knapsack, which leads to an APTAS for the LP relaxation. This suffices for our purposes here.

Finally, we show how to derandomize our general method.

2 Preliminaries

In all the packing problems considered in this paper we are given a set I of d -dimensional *items*, the i -th corresponding to a d -tuple $(t_i^1, t_i^2, \dots, t_i^d)$, that must be packed into the smallest number of unit-size *bins*, corresponding to the d -tuple $(1, \dots, 1)$. For the case $d = 1$, we let $s_i := t_i^1$ be the *size* of item i . For the case $d = 2$, for $i \in I$ we will write b_i for t_i^1 and h_i for t_i^2 . The first dimension will be called the *width* (or *basis*) and the second dimension will be called the *height*. Moreover, we will let $a_i := b_i \cdot h_i$ denote the *area* of item i .

For d -Dim Vector Packing, a set C of items can be packed into a bin if $\sum_{i \in C} t_i^j \leq 1$ for each $j = 1, \dots, d$. For d -Dim Bin Packing, the items are d -dimensional parallelepipeds with sizes given by the associated tuple, the bins are d -dimensional cubes, and a set C of items can be packed into a bin if the items can be placed in the bin without any two overlapping with each other. We only consider the

orthogonal packing case, where the items must be placed so that their edges are parallel to the edges of the bin. We address both the classical version without rotations, in which, for each coordinate, all edges associated with that coordinate in a bin have to be parallel, and the version with (orthogonal) rotations, in which this restriction is not imposed.

Given an instance I of a minimization problem, we let $\text{opt}(I)$ denote the value of the optimal solution of the problem for I . Given a (deterministic) algorithm for the problem, we say that it has *asymptotic approximation guarantee* ρ if there exists a constant δ such that the value of the solution found by the algorithm is at most $\rho \text{opt}(I) + \delta$ for each instance I . If $\delta = 0$, then the algorithm has (*absolute*) *approximation guarantee* ρ . Given a randomized algorithm for the problem, we say that it has asymptotic approximation guarantee ρ if there exists a constant δ such that the value of the solution found by the algorithm is at most $\rho \text{opt}(I) + \delta$ with a probability that tends to 1 as $\text{opt}(I)$ tends to infinity. An algorithm with an asymptotic approximation guarantee of ρ is called an *asymptotic ρ -approximation algorithm*. An APTAS is a family of polynomial-time algorithms such that, for each $\varepsilon > 0$, there is a member of the family with asymptotic approximation guarantee $1 + \varepsilon$. If $\delta = 0$ for every ε , then this is a PTAS.

All above problems could be formulated as the following general Set Covering problem, in which a set I of items has to be covered by *configurations* from the collection $\mathcal{C} \subseteq 2^I$, where each configuration $C \in \mathcal{C}$ corresponds to a set of items that can be packed into a bin:

$$\min \left\{ \sum_{C \in \mathcal{C}} x_C : \sum_{C \ni i} x_C \geq 1 \ (i \in I), \ x_C \in \{0, 1\} \ (C \in \mathcal{C}) \right\}. \quad (1)$$

As mentioned earlier, the collection \mathcal{C} is given implicitly since it is exponentially large for the applications we consider, and hence we need to specify how to solve the LP relaxation of (1). The dual of this LP is given by

$$\max \left\{ \sum_{i \in I} w_i : \sum_{i \in C} w_i \leq 1 \ (C \in \mathcal{C}), \ w_i \geq 0 \ (i \in I) \right\}. \quad (2)$$

Note that the separation problem for the dual is the following Knapsack-type problem: Given weights on items w_i , find, if any, a feasible configuration in which the total weight of items exceeds 1. By the well known connection between separation and optimization [15, 16, 26], we have that

Theorem 1 *If there exists a PTAS for the optimization version of the separation problem for (2), that is given $w_i \in \mathbb{R}_+^{|I|}$ solve $\max_{C \in \mathcal{C}} \sum_{i \in C} w_i$, then there exists a PTAS for the LP relaxation of (1).*

3 The General Method

Our method, hereafter called *Round and Approx* (R&A), constructs an approximate solution of the Set Covering problem (1) by performing the following steps, where $\alpha > 0$ is a parameter whose value will be specified later.

1. Solve the LP relaxation of (1) (possibly approximately). Let x^* be the (near-)optimal solution of the LP relaxation and $z^* := \sum_{C \in \mathcal{C}} x_C^*$ be its value;
2. Define the binary vector x^r starting with $x_C^r := 0$ for $C \in \mathcal{C}$ and then repeating the following for $\lceil \alpha z^* \rceil$ iterations: select one configuration C' at random, letting each $C \in \mathcal{C}$ be selected with probability x_C^*/z^* , and let $x_{C'}^r := 1$;

3. Consider the set of items $S \subseteq I$ that are not covered by x^r , namely $i \in S$ if and only if $\sum_{C \ni i} x_C^r = 0$, and the associated optimization problem for the *residual instance*:

$$\min \left\{ \sum_{C \in \mathcal{C}} x_C : \sum_{C \ni i} x_C \geq 1 \ (i \in S), \ x_C \in \{0, 1\} \ (C \in \mathcal{C}) \right\}. \quad (3)$$

Apply some approximation algorithm to problem (3) yielding solution x^a ;

4. Return the solution $x^h := x^r + x^a$.

Note that in Step 2 each selection is independent of the others (i.e., the same configuration may be selected more than once).

Of course, the quality of the final solution depends on the quality of the approximation algorithm used to solve the residual instance. Here we focus our attention on the case in which this latter quality can be expressed in terms of a small set of “weight” vectors in $\mathbb{R}^{|I|}$, as stated in Definition 1 below.

Given a set $S \subseteq I$, with a slight abuse of notation we let S denote also the Set Covering instance defined by the items in S . Moreover, we let $\text{opt}(S)$ and $\text{appr}(S)$ denote, respectively, the value of the optimal solution of (3) and the value of the heuristic solution produced by the approximation algorithm that we consider.

Below we define the class of the *subset-oblivious* algorithms, which are very useful for our analysis. Intuitively, since we apply a randomized rounding in Step 2, we do not know in advance which will be the subset S of remaining items, but we still want our approximation algorithm to perform “well” in a suitably-defined sense. The definition below formalizes the notion of “subset independence” that we need.

Definition 1 *An asymptotic ρ -approximation algorithm for problem (1) is called subset oblivious if, for any fixed $\varepsilon > 0$, there exist constants k , ψ and δ (possibly depending on ε) such that, for every instance I of (1), there exist vectors $w^1, \dots, w^k \in \mathbb{R}^{|I|}$ with the following properties:*

- (i) $\sum_{i \in C} w_i^j \leq \psi$, for each $C \in \mathcal{C}$ and $j = 1, \dots, k$;
- (ii) $\text{opt}(I) \geq \max_{j=1}^k \sum_{i \in I} w_i^j$;
- (iii) $\text{appr}(S) \leq \rho \max_{j=1}^k \sum_{i \in S} w_i^j + \varepsilon \text{opt}(I) + \delta$, for each $S \subseteq I$.

Property (i) says that the vectors obtained from w^1, \dots, w^k by dividing all the entries by constant ψ must be feasible for the dual of the LP relaxation of (1), Property (ii) provides a lower bound on the value of the optimal solution for the whole instance I , and Property (iii) guarantees that the value of the approximate solution on subset S is not significantly larger than ρ times the “fraction” of the lower bound in (ii) associated with S .

It is instructive to consider an example. Suppose we have an instance of 1-Dim Bin Packing and we consider the Next Fit algorithm, where each item is placed in the current bin if it fits, and placed in a new empty bin otherwise (closing the previous bin). We wish to show that Next Fit is an asymptotic 2-approximation subset-oblivious algorithm. To do this, we let $k := 1$ and define the vector w^1 by $w_i^1 := s_i$, the size of item i , for $i \in I$. Then clearly Property (i) is satisfied with $\psi = 1$, as no bin can contain items with total size more than 1. Property (ii) follows trivially as the number of bins used is at least equal to the total size of the items in the instance. Property (iii) holds with $\rho = 2$ and $\delta = 1$ (for any $\varepsilon \geq 0$), and follows by observing that the total size of the items in every two consecutive bins packed by Next Fit is at least 1.

In general there are many candidates for the vectors w^i . In particular, any feasible solution w to the dual problem defined by (2) satisfies Property (i) with $\psi = 1$, and satisfies Property (ii) by LP duality. Typically, the non-trivial part is to choose a small collection of appropriate vectors w^i and show that Property (iii) holds with a reasonable value of ρ .

Our main result is the following:

Theorem 2 *Suppose R&A uses an asymptotic μ -approximation algorithm to solve the LP relaxation in Step 1, an asymptotic ρ -approximation subset-oblivious algorithm for problem (1) in Step 3 (with $\mu < \rho$), and $\alpha := \ln(\rho/\mu)$ in Step 2. Then, for any fixed $\gamma > 0$, the cost of the final solution is at most*

$$(\mu(\ln(\rho/\mu) + 1) + \varepsilon) \text{opt}(I) + \delta + \gamma z^* + 1 \quad (4)$$

with probability at least $1 - k e^{-2(\gamma z^*)^2/(\psi^2 \lceil z^* \ln \rho \rceil)}$. In other words, for any fixed $\varepsilon > 0$, R&A is a randomized asymptotic $(\mu(\ln(\rho/\mu) + 1) + \varepsilon)$ -approximation algorithm for problem (1).

Corollary 1 *If R&A uses an APTAS to solve the LP relaxation in Step 1, then, for any fixed $\varepsilon > 0$, it is a randomized asymptotic $(\ln \rho + 1 + \varepsilon)$ -approximation algorithm for problem (1).*

We need the following concentration inequality in the analysis of R&A, due to McDiarmid [24] (see also [25] for a nice survey on concentration inequalities).

Lemma 1 (Independent Bounded Difference Inequality) *Let $X = (X_1, \dots, X_n)$ be a family of independent random variables, with $X_j \in A_j$ for $j = 1, \dots, n$, and $f : \prod_{j=1}^n A_j \rightarrow \mathbb{R}$ be a function such that*

$$|f(x) - f(x')| \leq c_j$$

whenever the vectors x and x' differ only in the j -th coordinate. Let $E(f(X))$ be the expected value of the random variable $f(X)$. Then, for any $t \geq 0$,

$$\Pr[f(X) - E(f(X)) \geq t] \leq e^{-2t^2/\sum_{j=1}^n c_j^2}.$$

Proof of Theorem 2 The cost of the rounded solution x^r produced in Step 2 is at most $\lceil \alpha z^* \rceil \leq \alpha \mu \text{opt}(I) + 1 = \mu \ln(\rho/\mu) \text{opt}(I) + 1$.

We now estimate the cost of x^a . Let S be the set of uncovered elements after Step 2. Note that S is a random set. Consider the random variable $\sum_{i \in S} w_i^j$ for $j = 1, \dots, k$. By the structure of the algorithm and linearity of expectation, we know that

$$E\left(\sum_{i \in S} w_i^j\right) = \sum_{i \in I} w_i^j \Pr(i \in S) = \sum_{i \in I} w_i^j \left(1 - \sum_{C \ni i} x_C^*/z^*\right)^{\lceil \alpha z^* \rceil} \leq e^{-\alpha} \sum_{i \in I} w_i^j,$$

where the last inequality follows from $\sum_{C \ni i} x_C^* \geq 1$ for $i \in I$ and $(1 - 1/a)^{\lceil \alpha a \rceil} \leq (1 - 1/a)^{\alpha a} \leq e^{-\alpha}$ for $a > 0$.

By the structure of the algorithm, the random variable $\sum_{i \in S} w_i^j$ is a function of $\lceil \alpha z^* \rceil$ independent random variables. Changing the value of any of these random variables may lead to the selection of a configuration C' in place of a configuration C . Letting S' be the resulting residual instance in the latter case, we have $|\sum_{i \in S} w_i^j - \sum_{i \in S'} w_i^j| \leq \max\{\sum_{i \in C \setminus C'} w_i^j, \sum_{i \in C' \setminus C} w_i^j\} \leq \psi$, by Property (i). Therefore, applying Lemma 1, we get

$$\Pr\left[\sum_{i \in S} w_i^j - E\left(\sum_{i \in S} w_i^j\right) \geq \gamma z^*\right] \leq e^{-2(\gamma z^*)^2/(\psi^2 \lceil \alpha z^* \rceil)}. \quad (5)$$

Using (5), the union bound on j , and Properties (ii) and (iii) of subset oblivious algorithms (Definition 1) we obtain that, for any constant $\gamma > 0$, the cost $\text{appr}(S)$ of the approximate solution x^a is at most

$$\begin{aligned} \rho \max_{j=1}^k \sum_{i \in S} w_i^j + \varepsilon \text{opt}(I) + \delta &\leq \rho e^{-\alpha} \max_{j=1}^k \sum_{i \in I} w_i^j + \varepsilon \text{opt}(I) + \delta + \gamma z^* \\ &\leq (\rho e^{-\alpha} + \varepsilon) \text{opt}(I) + \delta + \gamma z^* = (\mu + \varepsilon) \text{opt}(I) + \delta + \gamma z^* \end{aligned}$$

with probability at least $1 - k e^{-2(\gamma z^*)^2 / (\psi^2 \lceil \alpha z^* \rceil)}$. \square

In Section 8 we show how to derandomize the method.

In the rest of the paper, we represent the set-covering LP relaxation of the residual instance S as:

$$\min \left\{ \sum_{C \in \mathcal{C}} x_C : \sum_{C \ni i} x_C \geq 1 \ (i \in S), \ x_C \geq 0 \ (C \in \mathcal{C}) \right\} \quad (6)$$

and its dual as:

$$\max \left\{ \sum_{i \in S} w_i : \sum_{i \in C} w_i \leq 1 \ (C \in \mathcal{C}), \ w_i \geq 0 \ (i \in I) \right\}. \quad (7)$$

Note that the feasible region of (7) is independent of the choice of the subset S , which appears only in the objective. This observation will be crucial in defining subset-oblivious algorithms.

4 A Subset-Oblivious APTAS for 1-Dim Bin Packing

The structural property of 1-Dim Bin Packing proved in this section is the key to analyze versions of R&A for generalizations of the problem. Recall that for an instance I the size of an item $i \in I$ is denoted by s_i .

Lemma 2 *For any fixed $\varepsilon > 0$, there exists a polynomial-time asymptotic $(1+\varepsilon)$ -approximation subset-oblivious algorithm for 1-Dim Bin Packing.*

Proof We show that the APTAS of [13] with very minor modifications is a subset-oblivious algorithm. Let $\sigma := \varepsilon / (1 + \varepsilon)$, $M := \{i \in I : s_i < \sigma\}$ be the set of *small items* and $L := \{i \in I : s_i \geq \sigma\}$ the set of *large items*, with $\ell := |L|$, assuming $s_1 \geq s_2 \geq \dots \geq s_\ell$, i.e., items are ordered according to decreasing sizes.

Define the following *reduced sizes* for the items in L starting from their original *real sizes* s_1, \dots, s_ℓ . If $\ell < 2/\sigma^2$, we let $p := \ell$ and, $L_i := \{i\}$, $\underline{s}_i := s_i$ for $i = 1, \dots, \ell$, i.e., we do not change the sizes. Otherwise, using the fundamental *linear grouping* technique of [13], we define $q := \lfloor \ell \sigma^2 \rfloor$ and define $p := \lceil \ell/q \rceil$ groups L_1, \dots, L_p of consecutive items in L , where, for $j = 1, \dots, p-1$, L_j contains items $(j-1)q + 1, \dots, jq$, and L_p contains items $(p-1)q + 1, \dots, \ell$ (the smallest items in L). The reduced size \underline{s}_j of each item in group L_j is given by the size of the smallest item in the group, namely $\underline{s}_j := \min_{i \in L_j} s_i$. It is easy to check that $p \leq 1 + 3/\sigma^2 = O(1/\varepsilon^2)$.

For a given $S \subseteq I$, consider the following LP, which is the counterpart of (6) for reduced sizes, where items of the same size are associated with a unique constraint. Let c^1, \dots, c^m be the collection of non-negative integer vectors $c \in \{0, \dots, \lfloor 1/\sigma \rfloor\}^p$ such that $\sum_{j=1}^p c_j \underline{s}_j \leq 1$. These vectors represent the

feasible packing configurations of the items in L with reduced sizes. Note that $m = O(O(1/\varepsilon)^{O(1/\varepsilon^2)})$. The LP is:

$$\min \left\{ \sum_{r=1}^m x_r : \sum_{r=1}^m c_j^r x_r \geq |L_j \cap S| \ (j = 1, \dots, p), \ x_r \geq 0 \ (r = 1, \dots, m) \right\} \quad (8)$$

and its dual:

$$\max \left\{ \sum_{j=1}^p |L_j \cap S| v_j : \sum_{j=1}^p c_j^r v_j \leq 1 \ (r = 1, \dots, m), \ v_j \geq 0 \ (j = 1, \dots, p) \right\}. \quad (9)$$

We define the following approximate solution starting from an optimal basic solution x^* of LP (8). Consider the solution $\lceil x^* \rceil$ obtained by rounding up x^* . This corresponds to a feasible packing of the items in $L \cap S$ with reduced sizes (in case an item is packed into more bins, we keep it in only one of these bins). If no grouping was performed, this is also a feasible packing for the real sizes. Otherwise, we define the following packing for the real sizes: in the rounded solution, for $i \leq |L \cap S| - q$, use the space for the reduced size of the i -th largest item in $L \cap S$ to pack the real size of the $(i + q)$ -th largest item in $L \cap S$ (which is not larger by definition of the grouping procedure and since each group contains at most q items). The real sizes of the q largest items in $L \cap S$ are packed into q additional bins, one per bin. Finally, the small items in $M \cap S$ are packed in an arbitrary order by Next Fit, starting from the bins already containing some large items and considering a new bin only when the current small item does not fit in the current bin. Let $\text{appr}(S)$ be the value of the final solution produced.

We now show the subset obliviousness and approximation guarantee of the above algorithm. Note that the feasible region of the dual (9) does not depend on S . Moreover this feasible region is defined by p variables and m linear inequalities plus non-negativity conditions. Therefore, the number t of basic feasible solutions satisfies $t \leq \binom{p+m}{m}$, which is constant for fixed ε . This implies that, for all choices of the $2^{|I|}$ possible subsets S , the basic optimal solutions of (9) form a constant-size collection v^1, \dots, v^t .

We define the set of vectors w^1, \dots, w^k as follows, letting $k := t + 1$:

- for $f = 1, \dots, t$, we set $w_i^f := v_j^f$ for $j = 1, \dots, p$ and $i \in L_j$, and $w_i^f := 0$ for $i \in M$ (in other words, w^f is obtained by “expanding” the vector for reduced sizes v^f back to the actual sizes);
- $w_i^{t+1} := s_i$ for $i \in I$.

By the above definition, w^1, \dots, w^k are solutions of (7) (noting that also s is such a solution) and, for each $S \subseteq I$, $\max_{j=1}^{k-1} \sum_{i \in S} w_i^j$ is equal to the optimum of (9) for instance $L \cap S$ with reduced sizes. Moreover, $\text{opt}(S) \geq \sum_{i \in S} s_i = \sum_{i \in S} w_i^k$. Therefore, $\text{opt}(S) \geq \max_{j=1}^k \sum_{i \in S} w_i^j$ for each $S \subseteq I$. This implies Properties (i), with $\psi = 1$, and (ii) in Definition 1.

Finally, we show Property (iii) with $\delta = 1 + 3/\sigma^2$, completing the proof. If new bins are needed after packing the small items, we have that all the bins with the possible exception of the last one contain items for a total size of at least $(1 - \sigma)$. This implies

$$\text{appr}(S) \leq \frac{\sum_{i \in S} s_i}{1 - \sigma} + 1 = (1 + \varepsilon) \sum_{i \in S} w_i^k + 1$$

and we are done. On the other hand, if no new bins are needed for the small items, since the number of fractional components in the basic solution x^* is at most p we have that $\sum_{r=1}^m \lceil x_r^* \rceil \leq \sum_{r=1}^m x_r^* + p$. Moreover, recall that in case grouping is performed we use q additional bins for the q largest items, and note that $q \leq \ell \sigma^2 \leq \varepsilon \text{opt}(I)$, since $\sigma \leq \varepsilon$ and $\text{opt}(I) \geq \sigma \ell$ as all items in L have size at least σ . Now,

letting $w^* \in \{w^1, \dots, w^{k-1}\}$ be the dual solution of (7) corresponding to the optimal dual solution in $\{v^1, \dots, v^t\}$ of (9) associated with S , we have:

$$\begin{aligned} \text{appr}(S) &\leq \sum_{r=1}^m [x_r^*] + q \leq \sum_{r=1}^m x_r^* + 1 + 3/\sigma^2 + \varepsilon \text{opt}(I) \\ &= \sum_{i \in L \cap S} w_i^* + 1 + 3/\sigma^2 + \varepsilon \text{opt}(I) \leq \sum_{i \in S} w_i^* + \varepsilon \text{opt}(I) + 1 + 3/\sigma^2. \end{aligned}$$

□

It is interesting to note that the dependence of k on ε is multiply exponential.

5 Improved Approximation for d -Dim Vector Packing

We show how to combine the results of the previous sections to derive a polynomial-time randomized algorithm for d -Dim Vector Packing with asymptotic approximation guarantee arbitrarily close to $\ln d + 1$, which is $1.693\dots$ for $d = 2$. Recall that each item $i \in I$ corresponds to a 2-dimensional vector (b_i, h_i) .

Lemma 3 *For any fixed $\varepsilon > 0$, there exists a polynomial-time asymptotic $(d+\varepsilon)$ -approximation subset-oblivious algorithm for d -Dim Vector Packing, for constant d .*

Proof To avoid confusion, in this proof we will denote by $\text{opt}_{\text{BP}}(I)$ the value of the optimal 1-Dim Bin Packing solution for a generic instance I and $\text{appr}_{\text{BP}}(I)$ the value of the solution obtained by the subset-oblivious APTAS of Lemma 2 on instance I .

We give the proof in the case $d = 2$. The general case is proved analogously. Consider the following simple approximation algorithm analogous to the one in [13].

We partition the set I of items into sets $B := \{i \in I : b_i \geq h_i\}$ and $H := I \setminus B$, and for a given $S \subseteq I$ we pack the items in $B \cap S$ (resp., $H \cap S$) near-optimally into bins by applying the subset-oblivious APTAS of Lemma 2 to the Bin Packing instance with sizes $\{b_i : i \in B\}$ (resp., with sizes $\{h_i : i \in H\}$). Note that each feasible packing into one bin of one-dimensional items with sizes in $\{b_i : i \in B\}$ (resp., $\{h_i : i \in H\}$) corresponds to a feasible packing into one two-dimensional bin of the corresponding set of two-dimensional items from B (resp., H). Finally, we return the packing of the items in S defined by the bins in the two solutions obtained.

We now show that this algorithm is subset-oblivious. By Lemma 2, we have that for any $\zeta > 0$, there exist constants k_B, k_H, ξ and vectors $u^1, \dots, u^{k_B} \in \mathbb{R}^{|B|}$ and $v^1, \dots, v^{k_H} \in \mathbb{R}^{|H|}$ with the following properties:

$$\text{opt}_{\text{BP}}(B) \geq \max_{j=1}^{k_B} \sum_{i \in B} u_i^j, \quad \text{appr}_{\text{BP}}(B \cap S) \leq (1 + \zeta) \max_{j=1}^{k_B} \sum_{i \in B \cap S} u_i^j + \zeta \text{opt}(I) + \xi, \quad (10)$$

$$\text{opt}_{\text{BP}}(H) \geq \max_{j=1}^{k_H} \sum_{i \in H} v_i^j, \quad \text{appr}_{\text{BP}}(H \cap S) \leq (1 + \zeta) \max_{j=1}^{k_H} \sum_{i \in H \cap S} v_i^j + \zeta \text{opt}(I) + \xi. \quad (11)$$

Moreover, $u^1, \dots, u^{k_B} \in \mathbb{R}^{|B|}$ and $v^1, \dots, v^{k_H} \in \mathbb{R}^{|H|}$ are solutions of (2) for Bin Packing.

The required vectors w^1, \dots, w^k are the following, letting $k := k_B + k_H$:

- for $f = 1, \dots, k_B$, we set $w_i^f := u_i^f$ for $i \in B$, and $w_i^f := 0$ for $i \in H$;
- for $f = 1, \dots, k_H$, we set $w_i^{k_B+f} := 0$ for $i \in B$, and $w_i^{k_B+f} := v_i^f$ for $i \in H$.

Property (i) is trivially satisfied by the vectors w^1, \dots, w^k with $\psi = 1$. Moreover, Properties (ii) and (iii) are now simple to prove, letting $\zeta := \varepsilon/2$:

$$\text{opt}(I) \geq \max\{\text{opt}_{\text{BP}}(B), \text{opt}_{\text{BP}}(H)\} \geq \max\left\{\max_{j=1}^{k_B} \sum_{i \in B} w_i^j, \max_{j=1}^{k_H} \sum_{i \in H} v_i^j\right\} = \max_{j=1}^k \sum_{i \in I} w_i^j,$$

and, for each $S \subseteq I$, by (10) and (11) we obtain that

$$\begin{aligned} \text{appr}(S) &= \text{appr}_{\text{BP}}(B \cap S) + \text{appr}_{\text{BP}}(H \cap S) \\ &\leq (1 + \zeta) \max_{j=1}^{k_B} \sum_{i \in B \cap S} w_i^j + (1 + \zeta) \max_{j=1}^{k_H} \sum_{i \in H \cap S} v_i^j + 2\zeta \text{opt}(I) + 2\xi \\ &\leq 2(1 + \zeta) \max_{j=1}^k \sum_{i \in S} w_i^j + 2\zeta \text{opt}(I) + 2\xi. \end{aligned}$$

□

Since the separation problem for the dual of the configuration LP of the d -Dim Vector Packing is a maximum d -Dim (non-geometric) Knapsack Problem, which admits a PTAS for constant d [14], Theorem 1 implies the existence of a PTAS for the configuration LP for d -Dim Vector Packing. Thus, combining Lemma 3 and Theorem 2, we obtain:

Theorem 3 *For any fixed $\varepsilon > 0$, using a PTAS for the LP relaxation in Step 1 and the algorithm of Lemma 3 in Step 3, method R&A is a randomized polynomial-time asymptotic $(\ln(d + \varepsilon) + 1 + \varepsilon)$ -approximation algorithm for d -Dim Vector Packing, for constant d .*

In Table 1 we report the asymptotic approximation guarantees of R&A, of the simple approximation algorithm used in Step 3 of R&A, due to [13], and of the method by [8] for various values of d .

d	2	3	4	5	6
[13]	2	3	4	5	6
[8]	3	3.5	3.833	4.083	4.283
R&A	1.693	2.098	2.386	2.609	2.791

Table 1: Asymptotic approximation guarantees for d -Dim Vector Packing (ε omitted for brevity).

6 Improved Approximation for 2-Dim Bin Packing without Rotations

We now show the implications of our approach for 2-Dim Bin Packing, recalling that the items in an instance I correspond to rectangles with sizes $\{(b_i, h_i) : i \in I\}$. This case is much more involved than the one of the previous section. The subset-oblivious algorithm that we present is essentially copied from [3], and analogous to the approximation algorithm of [5], both having asymptotic approximation guarantee arbitrarily close to 1.691... The main difference between our algorithm and the one of [5] is that the latter is a very fast combinatorial algorithm combined with an APTAS for 1-Dim Bin Packing, whereas ours is based on the solution of a certain LP, which is useful to derive the vectors that we need to show subset obliviousness.

In the following sections, we will extensively use the *Next Fit Decreasing Height* (NFDH) procedure introduced by [9]. NFDH considers the items in decreasing order of height and greedily packs them in

this order into *shelves*. A set of items is said to be packed into a shelf if their bottom edges all lie on the same horizontal line. More specifically, starting from the bottom of the first bin, the items are packed left justified into a shelf until the next item does not fit. The shelf is then closed (i.e., no further item will be packed into the shelf), and the next item is used to define a new shelf whose bottom line touches the top edge of the tallest (first) item in the shelf below. If the shelf does not fit in the bin, i.e., if the next item does not fit on top of the tallest item in the shelf below, the bin is closed and a new bin is started. The procedure continues until all items are packed.

The key properties of NFDH that we need are:

Lemma 4 ([9]) *Given a set of items of height at most h , if NFDH is used to pack these items into a rectangle of width B and height H , and not all the items fit, letting \underline{B} be the minimum total width of the items in a shelf in the rectangle, then the total area not occupied in the rectangle is at most $hB + (B - \underline{B})H$.*

Lemma 5 ([9]) *Given a set of items S , the total number of bins used by NFDH to pack the items is at most $4 \sum_{i \in S} b_i \cdot h_i + 2$.*

Another standard tool in the design of bin packing algorithms is the so-called *harmonic transformation*, first introduced by Lee and Lee [23]. Let t be a positive integer and x be a positive real in $(0, 1]$. The harmonic transformation f_t with parameter t is defined as follows:

$$f_t(x) := 1/q \text{ if } x \in (1/(q+1), 1/q] \text{ for an integer } q \leq t-1; f_t(x) := tx/(t-1) \text{ if } x \in (0, 1/t].$$

The crucial property of this transformation is that, for any sequence x_1, \dots, x_n with $x_i \in (0, 1]$ for $i = 1, \dots, n$ and $\sum_{i=1}^n x_i \leq 1$, we have $\sum_{i=1}^n f_t(x_i) \leq \Pi_\infty + 1/(t-1)$. Here, $\Pi_\infty = 1.691\dots$ is the *harmonic constant* defined in [23]. In order to get rid of the inconvenient $1/(t-1)$ term, we will use a slight variant of f_t , called g_t , defined by

$$g_t(x) := f_t(x) \text{ if } x \in (1/t, 1]; g_t(x) := x \text{ if } x \in (0, 1/t]. \quad (12)$$

Lemma 6 *For any positive integer t and for any sequence x_1, \dots, x_n with $x_i \in (0, 1]$ for $i = 1, \dots, n$ and $\sum_{i=1}^n x_i \leq 1$, it holds that $\sum_{i=1}^n g_t(x_i) \leq \Pi_\infty$.*

Proof Since $g_t(x) \leq f_u(x)$ for every $x \in (0, 1]$ and for every positive integer $u \geq t$, we have $\sum_{i=1}^n g_t(x_i) \leq \lim_{u \rightarrow \infty} \sum_{i=1}^n f_u(x_i) \leq \Pi_\infty$. \square

Finally, we need a result that was the key to prove the approximation guarantee in [5], for which we restate an explicit (easy) proof for the sake of completeness:

Lemma 7 ([12, 27]) *Let S be a set of items with sizes $\{(b_i, h_i) : i \in S\}$ that fits into a bin and let w be an arbitrary dual feasible solution of (2) for the 1-Dim Bin Packing instance with sizes $\{h_i : i \in S\}$. Then the set of items with sizes $\{(b_i, w_i) : i \in S\}$ also fits into a bin.*

Proof Consider a feasible packing of the items in S with heights h_i into a bin. Consider the items in increasing order of distance of their bottom edge from the bottom edge of the bin (breaking ties arbitrarily). For each item i in this order, without changing the horizontal coordinates, first move i down as far as possible without overlapping other items and then change the height from h_i to w_i , greedily moving up some items if i overlaps with them after this increase.

If the corresponding packing is not feasible, there must be some sequence i_1, \dots, i_m of items that are on top of each other such that $\sum_{\ell=1}^m w_{i_\ell} > 1$. Given that we did not change the widths and the

horizontal coordinates, these items are on top of each other also in the initial packing, i.e., $\sum_{\ell=1}^m h_{i_\ell} \leq 1$, or, equivalently, the set $\{i_1, \dots, i_m\}$ is a feasible configuration in \mathcal{C} for the primal LP associated with (2) for the 1-Dim Bin Packing Problem. Given that w is a feasible solution of (2), we must have $\sum_{\ell=1}^m w_{i_\ell} \leq 1$, yielding a contradiction. \square

The main idea of [3] is the following: Consider the relaxation in which we wish to pack the items into a strip of height 1 and infinite width, with the goal of minimizing the total width. Clearly, if the items can be packed into $\text{opt}(I)$ bins, then they can also be packed into a strip of width $\text{opt}(I)$. In [3] it is shown that if the widths of items are *harmonic*, i.e., if all widths $\geq \varepsilon$ are of the form $1/q$ for some integer q , then there is an APTAS for strip packing with the additional property that, if we cut the strip vertically at some $x = i$ where i is an integer, then each item that is cut has width at most ε . This implies that this 2-Dim Strip Packing solution can be converted into a 2-Dim Bin Packing solution while increasing the cost by a factor of at most $1 + \varepsilon$. In other words, if the widths are harmonic, then there is an APTAS for 2-Dim Bin Packing. Thus an asymptotic Π_∞ approximation for general instances follows by first applying the harmonic transformation to the widths, increasing the optimal value by a factor of at most Π_∞ (see below for details), and then applying the above algorithm. The APTAS for harmonic widths with the special property mentioned above is based on a slight modification of the APTAS for strip packing due to [21]. In what follows, we give a self-contained proof of this result, presented in a way so that it is easy to show subset-obliviousness.

Lemma 8 *For any fixed $\varepsilon > 0$, there exists a polynomial-time asymptotic $(\Pi_\infty + \varepsilon)$ -approximation subset-oblivious algorithm for 2-Dim Bin Packing without rotations.*

Proof The proof is very similar in some parts to the subset oblivious APTAS for 1-Dim Bin Packing illustrated in Lemma 2. However, at the cost of some repetitions, we give full details also in this case to avoid possible confusion. Again, we introduce an internal parameter $\sigma > 0$, and show in the end that by defining appropriately σ as a function of ε we achieve the required accuracy.

Given the original instance I , first of all we apply the *harmonic transformation* (12) to the item widths, by defining $t := \lceil 1/\sigma \rceil$ and replacing each original width b_i with the *increased width* $\bar{b}_i := g_t(b_i)$. For the rest of the algorithm, we will only consider the increased widths of the items, called simply widths in the following, and forget about their original widths.

We let $M := \{i \in I : h_i < \sigma\}$ be the set of *short items* and $L := \{i \in I : h_i \geq \sigma\}$ be the set of *tall items*. Furthermore, we let $\ell := |L|$, $\bar{b}(L) := \sum_{i \in L} \bar{b}_i$ denote the total increased width of the tall items, and assume that these items are ordered according to decreasing heights. If $\bar{b}(L) < 2/\sigma^2$, we let $p := \ell$ and, $L_i := \{i\}$, $\underline{h}_i := h_i$ for $i = 1, \dots, \ell$. Otherwise, we define groups L_1, \dots, L_p of consecutive items in L , so that, for $j = 1, \dots, p-1$, the items in each group L_j have total width in $(\sigma^2 \bar{b}(L) - 1, \sigma^2 \bar{b}(L)]$, by inserting items in the group until the total width exceeds $\sigma^2 \bar{b}(L) - 1$, and the items in group L_p have total width at most $\sigma^2 \bar{b}(L)$. Note that we have $p \leq 2/\sigma^2$. We let the *reduced height* \underline{h}_j of each item in group L_j be the height of the shortest item in L_j .

Let c^1, \dots, c^m be the collection of the vectors $c \in \{0, \dots, \lfloor 1/\sigma \rfloor\}^p$ such that $\sum_{j=1}^p c_j \underline{h}_j \leq 1$. Note that such a c corresponds to a set of items that may be placed one on top of the other in a bin (with respect to the reduced heights), called a *slice*. Following the key observation in [21], if we allow items to be sliced vertically, for a given $S \subseteq I$ the resulting simplified 2-Dim Bin Packing problem is to assign widths to all possible slices so that, for each group L_j , the total width of the slices containing height \underline{h}_j is at least equal to the total width of the items in $L_j \cap S$. If we let x_r denote the width of slice c^r , we have the following LP, which is completely analogous to LP (8):

$$\min \left\{ \sum_{r=1}^m x_r : \sum_{r=1}^m c_j^r x_r \geq \sum_{i \in L_j \cap S} \bar{b}_i \ (j = 1, \dots, p), \ x_r \geq 0 \ (r = 1, \dots, m) \right\}. \quad (13)$$

We define the following approximate solution starting from an optimal basic solution x^* of LP (13), having at most p positive components. In fact, we start from a basic solution of the LP in which the “ \geq ” in the covering constraints is replaced by “ $=$ ”, which is easily seen to be equivalent to (13). Starting from such a solution, in which there are no useless spaces allocated for the items in L , simplifies both the description and the analysis of the algorithm. For each positive component x_r^* associated with vector c^r , we introduce $\lceil x_r^* \rceil$ bins in the solution. In each of these bins, we define c_j^r shelves of height \underline{h}_j , and use these shelves to pack the items as follows. The covering constraints in the LP ensure that the items in $L \cap S$ with their reduced heights can be packed into these shelves if they can be sliced vertically (packing slices into distinct shelves/bins). More precisely, such a packing can be obtained by considering, for $j = 1, \dots, p$, the shelves of height \underline{h}_j and packing the items in $L_j \cap S$ into these shelves in decreasing order of (original) height (i.e., in increasing order of index) – when an item does not fit, the slice that fits is packed into the current shelf and the other slice is packed into the next shelf. We call this the *sliced packing for the reduced heights*.

We then partition the items in $L \cap S$ into *new* groups $L'_1, \dots, L'_{p'}$ (again considering the items in decreasing order of original height), with $p' \leq 1/\sigma^2$, so that the total width of the items in each group is *exactly* $\sigma^2 \bar{b}(L)$, with the possible exception of the items in the last group, by possibly slicing vertically items between groups (note that slicing was not allowed in the definition of the old groups L_1, \dots, L_p). We put aside the items in L'_1 and, starting from the sliced packing for the reduced heights above, we define the *sliced packing for the original heights* by packing the items in each group L'_j with their original heights into the space used by the items in L'_{j-1} in the previous packing with reduced heights, for $j \geq 2$. Note that the definition of reduced heights guarantees that this packing is feasible. Moreover, this packing of items in L'_j is done in decreasing order of width, rather than in decreasing order of height.

After having defined the sliced packing for the original heights, in order to have a packing in which items are not sliced, the items in L'_1 together with all items that were sliced by forming the groups $L'_1, \dots, L'_{p'}$ or in defining the packing are packed into separate bins by NFDH.

Finally, we pack the short items in $M \cap S$ using the rectangles of width 1 and height $1 - \sum_{j=1}^p c_j^r \underline{h}_j$ that are left free for each bin associated with a positive component x_r^* in the LP, as well as additional bins if needed. The packing of these items is done by considering separately each *width class* (i.e., items with $\bar{b}_i = 1$, items with $\bar{b}_i = 1/2, \dots$, and items with $\bar{b}_i \leq 1/t$), and, for each class, packing the associated items by NFDH.

We now show the subset obliviousness and approximation guarantee of the above algorithm. By reasoning as in the proof of Lemma 2, we have a constant-size collection v^1, \dots, v^s of solutions of the dual of (13), and we define the set of vectors w^1, \dots, w^k as follows, letting $k := s + 1$:

- for $f = 1, \dots, s$, we set $w_i^f := (\bar{b}_i \cdot v_j^f) / \Pi_\infty$ for $j = 1, \dots, p$ and $i \in L_j$, and $w_i^f := 0$ for $i \in M$;
- $w_i^{s+1} := (\bar{b}_i \cdot h_i) / \Pi_\infty$ for $i \in I$.

Intuitively, each quantity w_i^f is an analog of the area of item $i \in I$, computed with respect to the increased widths (and the “modified” heights v_j^f for $f \leq s$) and scaled by the factor Π_∞ .

We first show that w^1, \dots, w^k are solutions of (7) for 2-Dim Bin Packing. By Lemma 6 we have that the vector $(\bar{b}_1 / \Pi_\infty, \bar{b}_2 / \Pi_\infty, \dots) = (g_t(b_1) / \Pi_\infty, g_t(b_2) / \Pi_\infty, \dots)$ is a feasible solution of (2) for the 1-Dim Bin Packing instance with one item of size b_i for $i \in I$. Therefore, by Lemma 7, for each set $C \subseteq \mathcal{C}$ of items that fit into a bin with their original sizes, the same items with the modified widths \bar{b}_i / Π_∞ also fit into a bin. This implies $\sum_{i \in C} (\bar{b}_i \cdot h_i) / \Pi_\infty \leq 1$. Moreover, if the items fit with their original heights, they also fit with their reduced heights. Then, by Lemma 7, given that v^f is a feasible

solution of the dual of (13), i.e., a feasible solution of (2) for the 1-Dim Bin Packing instance with $|L_j|$ items of size \underline{h}_j for $j = 1, \dots, p$, these items also fit if each (reduced) height \underline{h}_j is replaced by v_j^f . This implies $\sum_{i \in C} (\bar{b}_i \cdot v_j^f) / \Pi_\infty \leq 1$ for $f = 1, \dots, s$, and therefore Properties (i), with $\psi = 1$, and (ii) in Definition 1.

Finally, we show Property (iii). Let z^* be the optimal value of LP (13) and $w^* \in \{w^1, \dots, w^{k-1}\}$ be the vector corresponding to the optimal dual solution v^* of (13), for which

$$z^* = \sum_{j=1}^p \sum_{i \in L_j \cap S} \bar{b}_i \cdot v_j^* = \Pi_\infty \sum_{i \in L \cap S} w_i^* = \Pi_\infty \sum_{i \in S} w_i^*.$$

We have that the number of bins initially introduced is

$$\sum_{h=1}^m \lceil x_h^* \rceil \leq z^* + p \leq \Pi_\infty \sum_{i \in S} w_i^* + 2/\sigma^2.$$

We now bound the number of additional bins needed for the tall items. The structure of the widths guarantees that, in the sliced packing for the original heights, there are no items split in the shelves that contain only items from the same group L'_j having the same width value $1/q$ with $q \in \{1, \dots, t-1\}$. (This is the other key property of the harmonic transformation.) Accordingly, the total area of the tall items packed into additional bins is the sum of the following contributions:

- the total area of the items in L'_1 , which is at most $\sigma^2 \bar{b}(L) \leq 2\sigma^2 \sum_{i \in L} b_i \leq 2\sigma \text{opt}(I)$, where the first inequality follows from $\bar{b}_i \leq 2b_i$ for $i \in I$ (by definition of harmonic transformation) and the second from $\text{opt}(I) \geq \sum_{i \in L} b_i \cdot h_i \geq \sigma \sum_{i \in L} b_i$ (by definition of L);
- the total area of the items sliced by forming the groups, which is at most $p' \leq 1/\sigma^2$;
- the total area of the items split in the shelves that contain items from different groups L'_j and L'_{j+1} , which is at most $p' \leq 1/\sigma^2$ as these shelves are not more than p' due to the structure of the sliced packing for the reduced heights, which packs the items in decreasing order of height;
- the total area of the items split in the shelves that contain items from the same group L'_j but with distinct width classes, which is at most $t \cdot p' \leq \lceil 1/\sigma \rceil \cdot 1/\sigma^2$, as these shelves are not more than $t \cdot p'$ due to the structure of the sliced packing for the original heights, which packs the items in decreasing order of width;
- the total area of the items split in the shelves that contain items from the same group L'_j with widths in $(0, 1/t]$, which is at most $(1/t) \cdot \sum_{h=1}^m \lceil x_h^* \rceil \leq \sigma \Pi_\infty \sum_{i \in S} w_i^* + 2/\sigma$ since the area of the possible item split in each shelf is at most $1/t$ times the height of the shelf.

By Lemma 5, this implies that the number m_t of additional bins needed for the tall items is

$$m_t \leq 4 \cdot \left[2\sigma \text{opt}(I) + 2/\sigma^2 + \lceil 1/\sigma \rceil 1/\sigma^2 + \sigma \Pi_\infty \sum_{i \in S} w_i^* + 2/\sigma \right] + 2. \quad (14)$$

Therefore, if no additional bins are needed for the short items, we have

$$\text{appr}(S) \leq \sum_{r=1}^m \lceil x_r^* \rceil + m_t \leq \Pi_\infty \sum_{i \in S} w_i^* + 2/\sigma^2 + m_t. \quad (15)$$

If additional bins are needed for the short items, the structure of LP (13), recalling that we are solving the version with equality in the covering constraints, guarantees that the area of the items in $L \cap S$ is

$$\begin{aligned} \sum_{i \in L \cap S} \bar{b}_i \cdot h_i &\geq \sum_{j=1}^p \sum_{i \in L_j \cap S} \bar{b}_i \cdot h_j = \sum_{j=1}^p h_j \left(\sum_{i \in L_j \cap S} \bar{b}_i \right) = \sum_{j=1}^p h_j \left(\sum_{r=1}^m c_j^r x_r^* \right) \\ &\geq \sum_{r=1}^m \left(\sum_{j=1}^p c_j^r h_j \right) x_r^* \geq \sum_{r=1}^m \left(\sum_{j=1}^p c_j^r h_j \right) \lceil x_r^* \rceil - 2/\sigma^2. \end{aligned}$$

Moreover, with the exception of the last additional bin and the, at most t , bins containing short items associated with distinct width classes, all spaces occupied by the short items are nearly completely filled. More specifically, in all bins except at most $t + 1$, by Lemma 4 the area devoted to short items and not occupied is at most $\sigma + 1/t \leq 2\sigma$. Specifically:

- in the bins containing only short items having width $1/q$, $q \in \{1, \dots, t-1\}$, the total area not occupied by these items is at most σ (given that the total width of the items in each shelf is exactly 1),
- in the bins containing only short items having width $\leq 1/t$, the total area not occupied by these items is at most $\sigma + 1/t \leq 2\sigma$ (given that the total width of the items in each shelf is at least $1 - 1/t$).

Therefore, letting m_s denote the number of additional bins needed for the short items, the area of the items in $M \cap S$ is

$$\begin{aligned} \sum_{i \in M \cap S} \bar{b}_i \cdot h_i &\geq m_s(1 - 2\sigma) + \sum_{r=1}^m \lceil x_r^* \rceil \left(1 - \sum_{j=1}^p c_j^r h_j - 2\sigma \right) - (t + 1) \\ &\geq (1 - 2\sigma) \cdot \left[\sum_{r=1}^m \left(1 - \sum_{j=1}^p c_j^r h_j \right) \lceil x_r^* \rceil + m_s \right] - (1/\sigma + 2), \end{aligned}$$

and therefore

$$\sum_{i \in S} \bar{b}_i \cdot h_i \geq (1 - 2\sigma) \cdot \left[\sum_{r=1}^m \lceil x_r^* \rceil + m_s \right] - (1/\sigma + 2/\sigma^2 + 2).$$

Recalling that $\bar{b}_i \cdot h_i = \Pi_\infty w_i^k$ for $i \in I$, we have

$$\begin{aligned} \text{appr}(S) &= \sum_{r=1}^m \lceil x_r^* \rceil + m_s + m_t \leq \frac{\sum_{i \in S} \bar{b}_i \cdot h_i + (1/\sigma + 2/\sigma^2 + 2)}{(1 - 2\sigma)} + m_t \\ &\leq \frac{\Pi_\infty \sum_{i \in S} w_i^k + (1/\sigma + 2/\sigma^2 + 2)}{(1 - 2\sigma)} + m_t. \end{aligned} \tag{16}$$

Combining (14), (15) and (16) (and recalling Lemma 5), by defining σ appropriately we have in both cases $\text{appr}(S) \leq (\Pi_\infty + \varepsilon) \max_{j=1}^k \sum_{i \in S} w_i^j + \varepsilon \text{opt}(I) + O(1)$. \square

Recall that for 2-Dim Bin Packing we cannot use Theorem 1 as the separation problem for the dual of the LP relaxation is a 2-Dim (Geometric) Knapsack, for which the existence of a PTAS is open for $d = 2$. However, we can show that:

Theorem 4 *There exists an APTAS for the LP relaxation of (1) for 2-Dim Bin Packing with and without rotations.*

The proof of this result is rather long and technical, and the ideas required are orthogonal to those considered in this paper. Thus we describe this proof in a companion paper [2]. The main point is that, in order to obtain an APTAS for the LP relaxation, it is sufficient to have a PTAS for the special case of 2-Dim Knapsack in which, roughly speaking, the profits of the items are close to their areas.

Combining Lemma 8, Theorem 4 and Theorem 2 we obtain:

Theorem 5 *For any fixed $\varepsilon > 0$, using the APTAS of Theorem 4 in Step 1 and the algorithm of Lemma 8 in Step 3, method R&A is a randomized polynomial-time asymptotic $(\ln(\Pi_\infty + \varepsilon) + 1 + \varepsilon)$ -approximation algorithm for 2-Dim Bin Packing without rotations.*

7 Improved Approximation for 2-Dim Bin Packing with Rotations

In this section we design a polynomial-time (deterministic) subset-oblivious approximation algorithm for 2-Dim Bin Packing with (orthogonal) rotations with asymptotic approximation guarantee arbitrarily close to Π_∞ , improving on the previously-known 2. An interesting aspect of this algorithm is the fact that, for the first time in this paper, a set of vectors satisfying the requirements in Definition 1 are used for algorithmic purposes and not only for the sake of the analysis. Moreover, this algorithm can be plugged into the R&A method, leading to an asymptotic approximation guarantee arbitrarily close to $\ln \Pi_\infty + 1$. The results presented hold also for the case in which the bin size is not the same for both dimensions, and we address the case of unit square bins only for simplicity of presentation.

Lemma 9 *For any fixed $\varepsilon > 0$, there exists a polynomial-time asymptotic $(\Pi_\infty + \varepsilon)$ -approximation subset-oblivious algorithm for 2-Dim Bin Packing with rotations.*

Proof In the proof, given a 2-Dim Bin Packing instance I , we let $\text{opt}(I)$ denote the optimal value for the problem we consider, in which rotations are allowed, and $\text{opt}_{2\text{BP}}(I)$ the optimal value for the case in which rotations are not allowed.

Given an item subset $S \subseteq I$, a *rotation* of S is represented by a partition $S_N \cup S_R$ of S , where S_N is the subset of items that are not rotated and S_R the subset of items that are rotated. A trivial exponential-time algorithm with asymptotic approximation guarantee arbitrarily close to Π_∞ is the following: Given $S \subseteq I$, try all the $2^{|S|}$ rotations of S and, for each of them, apply the algorithm of Lemma 8 to the items rotated accordingly (the proof below shows that this trivial algorithm is subset oblivious). The key point of the polynomial-time version is to avoid trying all the rotations.

Let σ be an internal parameter depending on the required accuracy ε . Given the original instance I , we define the item set $\bar{I} := \{(b_i, h_i) : i \in I\} \cup \{(h_i, b_i) : i \in I\}$, corresponding to the union of the non-rotated and rotated items in I . We consider the 2-Dim Bin Packing (without rotations) instance \bar{I} , and apply Lemma 8 with input accuracy σ in a constructive way, explicitly computing the vectors $w^1, \dots, w^k \in \mathbb{R}^{2|I|}$ as in the proof of that lemma, called $\bar{w}^1, \dots, \bar{w}^k$ in this proof. For each $i \in I$ and $j = 1, \dots, k$, we let u_i^j be the component of \bar{w}^j associated with item i non-rotated and v_i^j be the component of \bar{w}^j associated with item i rotated, respectively.

After having computed $\bar{w}^1, \dots, \bar{w}^k$, given $S \subseteq I$, we find the rotation $S_N \cup S_R$ of S that approximately minimizes

$$\max_{j=1}^k \sum_{i \in S_N} u_i^j + \sum_{i \in S_R} v_i^j, \quad (17)$$

as illustrated below, and then apply the algorithm of Lemma 8 to this rotation.

We complete the description of the algorithm by showing how we find a near-optimal rotation with respect to (17). If a rotation is represented by binary variables y_i , $i \in S$, where $y_i = 1$ if $i \in S_N$ and $y_i = 0$ if $i \in S_R$, (17) is equivalent to the following Integer LP:

$$\min \left\{ z : z \geq \sum_{i \in S} u_i^j y_i + \sum_{i \in S} v_i^j (1 - y_i) \ (j = 1, \dots, k), \ y_i \in \{0, 1\} \ (i \in S) \right\}. \quad (18)$$

It is easy to show that this problem is weakly NP-hard and solvable in pseudo-polynomial time by dynamic programming (given that k is fixed). In our case we solve the associated LP relaxation, finding an optimal basic solution, and then return the integer solution obtained by rounding the fractional y variables arbitrarily.

We now show the subset obliviousness and approximation guarantee of the algorithm. Note that, given that Property (i) in Lemma 8 holds with $\psi = 1$ for vectors $\bar{w}^1, \dots, \bar{w}^k \in \mathbb{R}^{2|I|}$, we have that for each $S \subseteq \bar{I}$

$$\text{opt}_{2\text{BP}}(S) \geq \max_{j=1}^k \sum_{i \in S} \bar{w}_i^j. \quad (19)$$

Let $I_N^* \cup I_R^*$ be the rotation of I associated with the optimal solution of 2-Dim Bin Packing with rotations for I , where $\text{opt}(I) = \text{opt}_{2\text{BP}}(I_N^* \cup I_R^*)$ is the corresponding number of bins. We define the vectors $w^1, \dots, w^k \in \mathbb{R}^{|I|}$ as follows:

- for $j = 1, \dots, k$, we set $w_i^j := u_i^j$ for $i \in I_N^*$, and $w_i^j := v_i^j$ for $i \in I_R^*$.

In other words, we keep only the components of $\bar{w}^1, \dots, \bar{w}^k$ associated with the items rotated as in $I_N^* \cup I_R^*$. (Note that this definition of w^1, \dots, w^k is non-constructive.)

In order to show Property (i), consider an arbitrary feasible configuration $C \in \mathcal{C}$ for the case with rotations. Given that the total area of the items in C is at most one, by Lemma 5 the items in the rotation $C_N^* \cup C_R^*$ of C corresponding to $I_N^* \cup I_R^*$ can be packed into at most 6 bins (very rough estimate), which implies

$$\sum_{i \in C} w_i^j = \sum_{i \in C_N^*} u_i^j + \sum_{i \in C_R^*} v_i^j = \sum_{i \in C_N^* \cup C_R^*} \bar{w}_i^j \leq \text{opt}_{2\text{BP}}(C_N^* \cup C_R^*) \leq 6,$$

where the inequality is implied by (19), yielding Property (i) with $\psi = 6$.

As to Property (ii), we have

$$\text{opt}(I) = \text{opt}_{2\text{BP}}(I_N^* \cup I_R^*) \geq \max_{j=1}^k \left(\sum_{i \in I_N^*} u_i^j + \sum_{i \in I_R^*} v_i^j \right) = \max_{j=1}^k \sum_{i \in I} w_i^j,$$

where the inequality is again implied by (19).

Finally, we show Property (iii). Given $S \subseteq I$ and letting $S_N \cup S_R$ be the rotation found by the

algorithm and $S_N^* \cup S_R^*$ be the rotation of S corresponding to $I_N^* \cup I_R^*$, we have:

$$\begin{aligned}
\text{appr}(S) &\leq (\Pi_\infty + \sigma) \max_{j=1}^k \left(\sum_{i \in S_N} u_i^j + \sum_{i \in S_R} v_i^j \right) + \sigma \text{opt}_{2\text{BP}}(\bar{I}) + O(1) \\
&\leq (\Pi_\infty + \sigma) \left[\max_{j=1}^k \left(\sum_{i \in S_N^*} u_i^j + \sum_{i \in S_R^*} v_i^j \right) + k \right] + \sigma \text{opt}_{2\text{BP}}(\bar{I}) + O(1) \\
&= (\Pi_\infty + \sigma) \left[\max_{j=1}^k \sum_{i \in S} w_i^j + k \right] + \sigma \text{opt}_{2\text{BP}}(\bar{I}) + O(1) \\
&\leq (\Pi_\infty + \sigma) \left[\max_{j=1}^k \sum_{i \in S} w_i^j + k \right] + 8\sigma \text{opt}(I) + \sigma + O(1),
\end{aligned}$$

recalling that the input accuracy of the algorithm in Lemma 8 is σ . Here, the first inequality follows from Lemma 8. The second inequality follows from the fact that rotation $S_N \cup S_R$ is an approximate solution of the optimization problem (17): letting $F \subseteq S$ be the set of indices of the variables y_i that are fractional in this optimal basic solution of the LP relaxation of (18), we have $|F| \leq k$, so by rounding the solution value increases by at most k since each objective function coefficient is at most 1. Finally, the last inequality follows from $\text{opt}_{2\text{BP}}(\bar{I}) \leq 8\text{opt}(I) + 2$, in turn implied by the trivial bound $\sum_{i \in I} b_i \cdot h_i \leq \text{opt}(I)$, by $\sum_{i \in \bar{I}} b_i \cdot h_i = 2 \sum_{i \in I} b_i \cdot h_i$, and by $\text{opt}_{2\text{BP}}(\bar{I}) \leq 4 \sum_{i \in \bar{I}} b_i \cdot h_i + 2$ by Lemma 5. \square

Combining Lemma 9, Theorem 4 and Theorem 2 we obtain:

Theorem 6 *For any fixed $\varepsilon > 0$, using the APTAS of Theorem 4 in Step 1 and the algorithm of Lemma 9 in Step 3, method R&A is a randomized polynomial-time asymptotic $(\ln(\Pi_\infty + \varepsilon) + 1 + \varepsilon)$ -approximation algorithm for 2-Dim Bin Packing with rotations.*

8 Derandomization

In this section we present a deterministic variant of method R&A in which Step 2 is replaced by a greedy procedure that defines x^r guided by a suitable potential function.

Let ψ be the constant and w^1, \dots, w^k be the vectors in Definition 1 for the algorithm in Step 3. Moreover, let x^* be the (near-)optimal solution of the LP relaxation of (1) and $C_1, \dots, C_m \in \mathcal{C}$ be the configurations associated with the nonzero components of x^* , with $z^* = \sum_{\ell=1}^m x_{C_\ell}^*$. Roughly speaking, the proof of Theorem 2 says that if we select $\lceil \alpha z^* \rceil$ of these configurations randomly according to probabilities $x_{C_\ell}^*/z^*$, letting S be the set of uncovered items after the selection, then $\sum_{i \in S} w_i^j$ is concentrated around $e^{-\alpha} \sum_{i \in I} w_i^j$ for $j = 1, \dots, k$.

The deterministic variant is a greedy procedure that, at each iteration, selects a configuration from C_1, \dots, C_m . The key part is the ‘‘score’’ according to which this configuration is selected, which is defined as follows. Let σ be a (small) parameter such that $\sigma\psi < 1$ to be specified later. For an arbitrary set of items S , consider the potential function

$$\Phi(S) := \ln \left(\sum_{j=1}^k \exp \left(\sigma \sum_{i \in S} w_i^j \right) \right). \quad (20)$$

(To improve readability, in this section we will often use the notation $\exp(x)$ in place of e^x .) Our deterministic variant of Step 2 is:

2'. Define the binary vector x^r starting with $x_{C'}^r := 0$ for $C \in \mathcal{C}$ and $S := I$ (i.e., all items are uncovered) and then repeating the following for $\lceil \alpha z^*/(1 - \sigma\psi/2) \rceil$ iterations: select the configuration $C' \in \{C_1, \dots, C_m\}$ such that $\Phi(S \setminus C')$ is minimum and let $x_{C'}^r := 1$ and $S := S \setminus C'$.

That is, at each iteration in Step 2', we choose the configuration C' that causes the largest decrease in the potential function. To analyze this procedure, we need the following key property of the potential function, which says that $\Phi(S)$ "tracks" $\max_{j=1}^k \sum_{i \in S} w_i^j$ (the main quantity of interest to us) up to an additive error of $(\ln k)/\sigma$.

Lemma 10 *For any arbitrary set of items S , the quantity $\Phi(S)/\sigma$ lies in the range*

$$\left[\max_{j=1}^k \sum_{i \in S} w_i^j, \max_{j=1}^k \sum_{i \in S} w_i^j + \ln k/\sigma \right] \quad (21)$$

Proof The result follows directly by the following two inequalities:

$$\sigma \max_{j=1}^k \sum_{i \in S} w_i^j = \ln \left(\exp \left(\sigma \max_{j=1}^k \sum_{i \in S} w_i^j \right) \right) \leq \ln \left(\sum_{j=1}^k \exp \left(\sigma \sum_{i \in S} w_i^j \right) \right) = \Phi(S)$$

and

$$\Phi(S) = \ln \left(\sum_{j=1}^k \exp \left(\sigma \sum_{i \in S} w_i^j \right) \right) \leq \ln \left(k \cdot \exp \left(\sigma \max_{j=1}^k \sum_{i \in S} w_i^j \right) \right) = \sigma \max_{j=1}^k \sum_{i \in S} w_i^j + \ln k.$$

□

We now show the main result of this section.

Lemma 11 *At the end of Step 2',*

$$\max_{j=1}^k \sum_{i \in S} w_i^j \leq (2 \ln k)/\sigma + e^{-\alpha} \max_{j=1}^k \sum_{i \in I} w_i^j.$$

Proof Let $\beta := (1 - \sigma\psi/2)/z^* < 1$ and let S_h denote the set of uncovered items after h iterations, with $S_0 = I$. We now show by a probabilistic argument that, for any $h \geq 0$,

$$\Phi(S_{h+1}) \leq \beta \ln k + (1 - \beta)\Phi(S_h). \quad (22)$$

Let $c_{\ell,h}^j := \sum_{i \in C_\ell \cap S_h} w_i^j$ be the sum of the components of vector w^j over the items in C_ℓ that are still uncovered after iteration h . Suppose at iteration $h + 1$, instead of following Step 2', we choose configuration C' as in the original Step 2, at random with probabilities $p_\ell := x_{C_\ell}^*/z^*$ for $\ell = 1, \dots, m$.

Considering the random variable $\exp\left(\sigma \sum_{i \in S_{h+1}} w_i^j\right)$, we have that

$$\begin{aligned} E\left(\exp\left(\sigma \sum_{i \in S_{h+1}} w_i^j\right)\right) &= \sum_{\ell=1}^m p_\ell \exp\left(\sigma \left(\sum_{i \in S_h} w_i^j - c_{\ell,h}^j\right)\right) \\ &= \exp\left(\sigma \sum_{i \in S_h} w_i^j\right) \left(\sum_{\ell=1}^m p_\ell \exp(-\sigma c_{\ell,h}^j)\right) \\ &\leq \exp\left(\sigma \sum_{i \in S_h} w_i^j\right) \left(\sum_{\ell=1}^m p_\ell (1 - \sigma c_{\ell,h}^j + (\sigma c_{\ell,h}^j)^2/2)\right) \end{aligned} \quad (23)$$

$$\leq \exp\left(\sigma \sum_{i \in S_h} w_i^j\right) \left(1 - \sum_{\ell=1}^m p_\ell (1 - \psi\sigma/2)\sigma c_{\ell,h}^j\right) \quad (24)$$

$$\leq \exp\left(\sigma \sum_{i \in S_h} w_i^j\right) \left(1 - (1 - \psi\sigma/2)\sigma \sum_{i \in S_h} w_i^j / z^*\right) \quad (25)$$

$$= \exp\left(\sigma \sum_{i \in S_h} w_i^j\right) \left(1 - \beta\sigma \sum_{i \in S_h} w_i^j\right)$$

$$\leq \exp\left(\sigma \sum_{i \in S_h} w_i^j\right) \cdot \exp\left(-\sigma\beta \sum_{i \in S_h} w_i^j\right) \quad (26)$$

$$= \exp\left(\sigma(1 - \beta) \sum_{i \in S_h} w_i^j\right).$$

Here, inequality (23) follows as $e^{-a} \leq 1 - a + a^2/2$ for $a < 1$, noting that $\sigma c_{\ell,h}^j \leq 1$. Inequality (24) follows as $\sum_{\ell=1}^m p_\ell = 1$ and $c_{\ell,h}^j \leq \psi$ by Property (i) of subset oblivious algorithms in Definition 1. Inequality (25) follows from the definition of p_ℓ and by observing that

$$\sum_{\ell=1}^m x_{C_\ell}^* c_{\ell,h}^j = \sum_{\ell=1}^m x_{C_\ell}^* \sum_{i \in C_\ell \cap S_h} w_i^j = \sum_{i \in S_h} w_i^j \sum_{C_\ell \ni i} x_{C_\ell}^* \geq \sum_{i \in S_h} w_i^j$$

as x^* is a feasible solution of the LP relaxation of (1). Finally, (26) follows as $e^{-a} \geq 1 - a$ for all $a \geq 0$.

By linearity of expectation, if we consider the deterministic Step 2', since the configuration C' is chosen to minimize $\Phi(S)$, we have:

$$\begin{aligned} \sum_{j=1}^k \exp\left(\sigma \sum_{i \in S_h \setminus C'} w_i^j\right) &\leq E\left(\sum_{j=1}^k \exp\left(\sigma \sum_{i \in S_{h+1}} w_i^j\right)\right) \\ &= \sum_{j=1}^k E\left(\exp\left(\sigma \sum_{i \in S_{h+1}} w_i^j\right)\right) \leq \sum_{j=1}^k \exp\left(\sigma(1 - \beta) \sum_{i \in S_h} w_i^j\right). \end{aligned} \quad (27)$$

We now use Holder's inequality, which states that for any non-negative a_1, \dots, a_k and b_1, \dots, b_k

and $p, q > 0$ such that $1/p + 1/q = 1$,

$$\sum_{j=1}^k a_j b_j \leq \left(\sum_{j=1}^k a_j^p \right)^{1/p} \cdot \left(\sum_{j=1}^k b_j^q \right)^{1/q}.$$

Setting $a_j = 1$, $b_j = \exp\left(\sigma(1-\beta) \sum_{i \in S_h} w_i^j\right)$, $p = 1/\beta$ and $q = 1/(1-\beta)$, we obtain that

$$\sum_{j=1}^k \exp\left(\sigma(1-\beta) \sum_{i \in S_h} w_i^j\right) \leq k^\beta \cdot \left(\sum_{j=1}^k \exp\left(\sigma \sum_{i \in S_h} w_i^j\right) \right)^{(1-\beta)}. \quad (28)$$

Combining (27) and (28), and taking logarithms on both sides, we have that

$$\ln \sum_{j=1}^k \exp\left(\sigma \sum_{i \in S_h \setminus C'} w_i^j\right) \leq \ln \sum_{j=1}^k \exp\left(\sigma(1-\beta) \sum_{i \in S_h} w_i^j\right) \leq \beta \ln k + (1-\beta) \ln \sum_{j=1}^k \exp\left(\sigma \sum_{i \in S_h} w_i^j\right)$$

which is precisely (22) since for the deterministic version $\Phi(S_{h+1}) = \Phi(S_h \setminus C')$.

By (22), for any $h \geq 0$ we have

$$\Phi(S_{h+1}) \leq \left(\beta \sum_{g=0}^h (1-\beta)^g \right) \ln k + (1-\beta)^{h+1} \Phi(I) \leq \ln k + (1-\beta)^{h+1} \Phi(I).$$

For $h+1 = \lceil \alpha z^*/(1-\sigma\psi/2) \rceil = \lceil \alpha/\beta \rceil$, using (21), we obtain

$$\begin{aligned} \max_{j=1}^k \sum_{i \in S} w_i^j &\leq (1/\sigma) \Phi(S_{h+1}) \leq (\ln k)/\sigma + (1-\beta)^{\lceil \alpha/\beta \rceil} \Phi(I)/\sigma \leq (\ln k)/\sigma + (1-\beta)^{\alpha/\beta} \Phi(I)/\sigma \\ &\leq (\ln k)/\sigma + e^{-\alpha} \Phi(I)/\sigma \leq (\ln k)/\sigma + e^{-\alpha} \left(\ln k/\sigma + \max_{j=1}^k \sum_{i \in I} w_i^j \right) \\ &\leq (2 \ln k)/\sigma + e^{-\alpha} \max_{j=1}^k \sum_{i \in I} w_i^j, \end{aligned}$$

proving the lemma. \square

Lemma 11 leads to the following variant of Theorem 2 (recalling that the only interesting case is the one in which $\ln \rho > 0$):

Theorem 7 *Consider the deterministic variant of R&A with Step 2' in place of Step 2. Suppose this variant uses an asymptotic μ -approximation algorithm to solve the LP relaxation in Step 1, an asymptotic ρ -approximation subset-oblivious algorithm for problem (1) in Step 3 (with $\mu < \rho$), and $\alpha := \ln(\rho/\mu)$ and $\sigma := (2\varepsilon/\ln \rho)/(\psi + \psi\varepsilon/\ln \rho)$ in Step 2'. Then, the cost of the final solution is at most*

$$\mu(\ln(\rho/\mu) + 1 + 2\varepsilon) \text{opt}(I) + \delta + (2 \ln k)(\psi + \psi\varepsilon/\ln \rho)/(2\varepsilon/\ln \rho) + 1. \quad (29)$$

In other words, for any fixed $\varepsilon > 0$, this variant is a deterministic asymptotic $\mu(\ln(\rho/\mu) + 1 + \varepsilon)$ -approximation algorithm for problem (1).

9 Final Remarks

Our general method for Set Covering is based on a very simple algorithmic idea, namely to use randomized rounding of the solution of the LP relaxation to find a partial cover, and then a suitable approximation algorithm to complete this partial cover. The reason why the method cannot be boosted, i.e., we cannot use R&A itself as approximation algorithm in Step 3, is that it is not clear whether R&A is subset oblivious itself, at least for the applications that we considered.

Given the simplicity of our method, it is natural to wonder if it can be applied to problems that are not variants of Bin Packing. However, we are not aware of any such applications. One obstacle against the use of R&A for other problems is the difficulty in deriving subset-oblivious algorithms (or proving that existing algorithms are subset oblivious).

Another interesting open problem is to apply our framework to d -Dim Bin Packing for $d \geq 3$. The key bottleneck here is to find good approximation algorithms to solve the LP relaxation of (1). For instance, the existence of a polynomial-time algorithm with an asymptotic approximation guarantee polynomial in d for this LP relaxation would lead to a polynomial-time algorithm with asymptotic approximation guarantee polynomial in d for the original problem, a significant improvement over the currently known guarantees, which are exponential in d .

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