MODULATED FILTER BANKS AND WAVELETS - A GENERAL UNIFIED THEORY

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ABSTRACT

This paper generalizes and unifies well-known results on modulated filter banks (MBFs) and modulated wavelet tight frames (MWTFTs). It classifies MBFs based on the discrete cosine or sine transforms that they are associated with. By proper choice of the form of modulation the PR perfect reconstruction (PR) conditions are seen to be (surprisingly) identical for all classes of MBFs. This has the interesting consequence that optimal MBF prototype designs can be shared across MFB classes. For some classes of MFBs associated MWTFTs do not exist, while for others they do. The results cover both orthogonal and biorthogonal MBFs, and the filters could be arbitrary sequences in $l^2(\mathbb{Z})$.

1. INTRODUCTION

A recent paper (see [1, 2]) consolidated and extended well-known results on modulated filter banks [5, 3, 6, 7]. These MBFs

1. have filters with non-overlapping ideal frequency responses
2. are associated with DCT III/IV (or equivalently DST III/IV) in their implementation
3. and do not allow for linear phase filters (even though the prototypes could be linear phase).

In trying to overcome 3, Lin and Vaidyanathan introduced a new class of linear-phase modulated filter banks by giving up 1 and 2 [4]. The author noticed that this class of filter banks was associated with DCT/DST I or II depending on whether $M$ (in this case half the number of channels in their filter banks) was even or odd. This paper was born out of wondering what types of MBFs could be associated with the four types of DCT and DST.

In a classical MFB the modulation sequence of analysis and synthesis filters depend on a parameter $\alpha \in \{M - 1, M - 2\}$ giving rise to Type 1 and Type 2 MBFs. Each MFB is associated with $J$ two channel PR filter banks constructed from the prototype filters, where

$$ J = \begin{cases} \frac{M}{2} & \text{Type 1, } M \text{ even} \\ \frac{M-1}{2} & \text{Type 1, } M \text{ odd} \\ \frac{M-3}{2} & \text{Type 2, } M \text{ even} \\ \frac{M-1}{2} & \text{Type 2, } M \text{ odd}. \end{cases} $$

Let

$$ R(M) = \{0, 1, \ldots, N - 1 \}, $$

and

$$ h_n = \begin{cases} \frac{1}{2} & n \in \{0, M\} \\ 0 & \text{otherwise}. \end{cases} $$

2. MODULATED FILTER BANKS

Two broad classes of MBFs (that together are associated with all four DCT/DSTs [7]) can be defined.

2.1. DCT/DST I/II based $2M$ Channel Filter Bank

$$ h_i(n) = k_i(n) \cos \left( \frac{\pi}{M} i(n - \frac{\alpha}{2}) \right), \quad i \in S_1 $$

$$ g_i(n) = k_i(n - M) \sin \left( \frac{\pi}{M} i(n - \frac{\alpha}{2}) \right), \quad i \in S_2 $$

$$ g_{M-i}(n) = k_i(n) \cos \left( \frac{\pi}{M} i(n + \frac{\alpha}{2}) \right), \quad i \in S_1 $$

$$ g_{M-i}(n) = -k_i(n + M) \sin \left( \frac{\pi}{M} i(n + \frac{\alpha}{2}) \right), \quad i \in S_2 $$

The sets $S_1$ and $S_2$ are defined depending on the parity of $\alpha$ as shown in Table 1. When $\alpha$ is even (i.e., Type 1 with odd

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$S_1$</th>
<th>$S_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>even, DCT/DST I</td>
<td>$R(M) \cup {M}$</td>
<td>$R(M)$</td>
</tr>
<tr>
<td>odd, DCT/DST II</td>
<td>$R(M)$</td>
<td>$R(M)$</td>
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Table 1. Class A MFB: the filter index sets $S_1$ and $S_2$.

2.2. DCT/DST III/IV based $M$-channel filter bank

$$ h_i(n) = h(n) \cos \left( \frac{\pi}{M} (i + 1)(n - \frac{\alpha}{2}) \right), \quad i \in R(M) $$

$$ g_i(n) = g(n) \cos \left( \frac{\pi}{M} (i + 1)(n + \frac{\alpha}{2}) \right), \quad i \in R(M) $$

DCT/DST I or DCT/DST II $2M$ channel filter banks will be called real DFT (or class A modulated) filter banks because the ideal responses correspond to that of an $M$-channel complex DFT filter bank. In fact they can be precisely interpreted as a pre-processed DFT. The DCT/DST III/IV based filter banks will be called class B modulated filter banks. With this terminology one sees that most results in the literature pertain to Class B, Type 1 modulated filter banks (with the exception of [4] which deals with some Class A Type 1 and Type 2 MBFs).

3. MFB PR CONDITIONS

A filter bank is PR iff

$$ \sum_n \sum_i h_i(Mn + n_1)g_i(-Mn - n_2) = \delta(n_1 - n_2), $$

or equivalently

$$ \sum_n h_i(n)g_i(-Ml - n) = \delta(l)\delta(i - j). $$
One might wonder how these two equivalent characterizations come about: its no more or less profound that the equivalence of matrix equations $AB = I$ and $BA = I$. Eqn. 7 captures biorthogonality constraints between the analysis and synthesis filters (very useful in wavelet theory) while Eqn. 6 captures the fact all channels put together have information to reconstruct the signal. Eqn. 6 is particularly useful in the MFB case since the (trigonometric) sum over $i$ can be simplified - PR conditions become explicit constraints on the prototype filters. PR filter banks are said to be unitary if the analysis and synthesis filters are time-reversals of each other, i.e., $g_i(n) = h_i(-n)$.

It can be shown that for PR it is necessary that $\alpha$, the modulation phase, is an integer [2]. Notice that the lengths of $h_i(n)$ and $g_i(n)$ do not appear in the modulation and hence the results hold for arbitrary sequences/filters (i.e., FIR/IIR or rational/irrational transfer function types).

Let
$$H(z) = \sum_{l=0}^{M-1} z^{-l} P_l(z^M) = \sum_{l=0}^{M-1} z^{-l} (P_{l,0}(z^{2M}) + z^{-M} P_{l,1}(z^{2M})).$$

$$G(z) = \sum_{l=0}^{M-1} z^l Q_l(z^M) = \sum_{l=0}^{M-1} z^l (Q_{l,0}(z^{2M}) + z^{2M} Q_{l,1}(z^{2M})).$$

and let
$$P(z) = \begin{bmatrix} P_{0,0}(z) & P_{1,1}(z)
F_{0,-1}(z) & F_{1,-1}(z) \end{bmatrix}$$

with $Q(z)$ defined similarly. Let $I$ be the 2x2 identity matrix.

**Theorem 1 (Modulated Filter Banks PR Theorem)** A modulated filter bank is PR iff
$$P(z)Q^T(z) = \frac{2}{M}I.$$

and furthermore if $\alpha$ is even $P_{\alpha,0}(z)Q_{\alpha,0}(z) = \frac{2}{M}I$. In the Type 2 case, we further require $P_{M-\alpha-1}(z)Q_{M-\alpha-1}(z) = \frac{2}{M}I$.

The result says that $P_l, F_{\alpha-1}, Q_l$ and $Q_{\alpha-1}$ form analysis and synthesis filters of a two channel PR filter bank (Eqn. 6 in Z-transform domain). All modulated filter banks above are unitary if they are PR and $g_i(n) = h_i(-n)$. For unitary modulated filter banks the PR conditions can be further simplified using the above condition which essentially means $P(z) = Q(z^{\alpha})$.

Indeed
$$P(z)P^T(z^{-1}) = \frac{2}{M}I.$$

This condition is equivalent to requiring $P_l$ and $P_{\alpha-1}$ being analysis filters of a two-channel unitary filter bank. Equivalently, for $l \in \mathbb{R}(M)$, $P_{0,0}$ and $P_{1,1}$ are power-complementary.

**Corollary 1 (Unitary MFB PR Theorem)** A modulated filter bank is unitary if $P_{0,0}(z)$ and $P_{1,1}(z)$ are power-complementary.

$$P_{0,0}(z)P_{0,0}(z^{-1}) + P_{1,1}(z)P_{1,1}(z^{-1}) = \frac{2}{M}, \quad l \in \mathbb{R}(M)$$

and furthermore if $\alpha$ is even $P_{\alpha,0}(z)P_{\alpha,0}(z^{-1}) = \frac{2}{M}$.

For the prototype filter of a unitary MFB to be linear phase it is necessary that
$$P_{\alpha-1}(z) = z^{-2k+1}P(z^{-1}),$$

for some integer $k$. In this case the prototype filter (if FIR) is of length $2Mk$ and symmetric about $(Mk - \frac{1}{2})$ in the Type 1 case and of length $2Mk - 1$ and symmetric about $(Mk - 1)$ (for both Class A and Class B MBFs). In the FIR case one can obtain linear-phase prototype filters by using the lattice parameterization [9] of two-channel unitary filter banks. Filter banks with FIR linear-phase prototype filters will be said to be canonical. In this case $P(z)$ is typically a (causal wlog) filter of length $2k$ for all $l$. For canonical modulated filter banks one has to check power complementarity only for $l \in \mathbb{R}(J)$.

If the prototype filter is linear phase then for Class A canonical unitary MFBs the MFB filters are also linear phase - seen from straightforward algebraic verification. This is not true for Class B MFBs. In fact, for Class B MBFs one can prove that linear phase MFB filters cannot be obtained [2].

Notice that since the only condition is pairs of polyphase components have to form two-channel PR filter banks (which could be arbitrary) one can construct many types of MFBs, with filters of arbitrary length etc.

### 4. DESIGN

The prototype filter for all the modulated filter banks can be designed by approximating an ideal lowpass filter with passband in $(-\pi/kM, \pi/kM)$. For Class A and Class B MBFs one expects Type 1 and Type 2 designs. However, the same design gives simultaneously optimal M-channel Class B MFB and an optimal 2M-channel Class A MFB. Fig. 1 and Fig. 2 give Type 1 and Type 2 examples respectively. Notice that in the Type 1 case DCT/DST I is involved while in the Type 2 case DCT/DST II is involved. This will reverse if $M$ had been even.

### 5. MODULATED WAVELET BASES

An $M$-band unitary filter bank gives rise to a wavelet tight frame (WTF) iff the linear condition $\sum_{n} h_o(n) = \sqrt{M}$. Class B MBFs give rise to WTFs iff for $l \in \mathbb{R}(J)$:

$$\Theta_l = \frac{\pi}{M} + \frac{\pi}{2M} \left( \frac{\alpha - 1}{2} \right).$$

\[\text{Figure 1. Type 1 MFB Freq. Resp. (in dB): M=19, N=2Mk=114 (a) Cosine Bank - Class A (b) Sine Bank - Class A (c) Class B}\]
With this parameterisation we define \( \Theta_1 \) according to Eqn. 13, where in the FIR case \( \Theta_1 = \sum_{k=0}^{N-1} d_{1,k} \) as before. Type 1 Class A MFBs give rise to a WTF iff \( \Theta_1 = \frac{\pi}{4} \) for all \( l \in R(J) \).

**Theorem 2 (Modulated Wavelet Tight Frames Theorem)**

A class A MFB of Type 1 gives rise to a WTF iff \( \Theta_1 = \frac{\pi}{4} \). A class B MFB (Type 1 or Type 2) gives rise to a WTF iff \( \Theta_1 = \frac{\pi}{4} + \frac{\pi}{2M} \left( \frac{2^m}{2} - 1 \right) \).

6. DO REAL DFT BASED \( 2M + 1 \) CHANNEL FILTER BANKS EXIST?

Notice that when \( N = 2M \) and the prototype filter is a constant, the Class A filter bank is essentially computing the real and imaginary parts of the \( 2M \) point DFT of a signal blocked into segments of length \( 2M \) and retaining relevant numbers taking into account the symmetry (in general when the prototype filter is not constant it is a windowed DFT). When \( N = 2Mk \) one therefore has a preprocessed DFT (where the even and odd components of a block of data are replaced by the “lowpass” and “highpass” components of corresponding points over several blocks). Can one not do the same with blocks of size \( 2M + 1 \)? Unfortunately, in this case there is no nontrivial solution for the prototype filter. However, when \( N = 2M + 1 \), one does get a \( 2M + 1 \) channel filter bank provided by the real and imaginary parts of the DFT basis functions.

**A PROOFS OF PR PROPERTY**

Proofs are based on the elementary trigonometric identity:

\[
\sum_{i=1}^{M-1} \cos(is) = \begin{cases} \frac{M}{2} & \text{if } s \in 2\pi \mathbb{Z} \\ \frac{1}{2} \left( \frac{\sin \left( \frac{2M-1}{2} \right)}{\sin \left( \frac{s}{2} \right)} - 1 \right) & \text{otherwise.} \end{cases}
\]

Let
\[
\zeta = \sum_{n} h(2Mn + n_1)g(-2Mn - n_2),
\]
and
\[
\eta = \sum_{n} h(2Mn + n_1 - M)g(-2Mn - n_2 + M),
\]
then
\[
\zeta + \eta = \sum_{n} h(Mn + n_1)g(-Mn - n_2),
\]
and
\[
\zeta - \eta = \sum_{n} (-1)^n h(Mn + n_1)g(-Mn - n_2).
\]

Eqn. 6, Eqs. 4a-4d and Eqs. 19-20 is used to infer the results in this paper. The MWTF property for Class A MFBs is also inferred algebraically by direct substitution.

1.1. Class A - DCT/DST I based

Let \( \hat{n} = n_1 - n_2 \), \( \hat{n} = n_1 + n_2 + \alpha \) and \( s = \frac{\pi}{M} \). For PR

\[
\frac{1}{2} \left( [a(\hat{n}) + a(\hat{n})] \zeta + [b(\hat{n}) - b(\hat{n})] \eta \right) = \delta(n_1 - n_2)
\]

where
\[
a(n) = \sum_{k=0}^{M} k_i \cos(isn) = \begin{cases} M + 1 & n \in 2MZ \\ 0 & \text{otherwise} \end{cases}
\]
and
\[
b(n) = \sum_{i=1}^{M-1} k_i \cos(isn) = \begin{cases} M - 1 & n \in 2MZ \\ -\frac{1}{2} (1 + (-1)^n) & \text{otherwise} \end{cases}
\]

There are four cases to consider depending on whether \( \hat{n} \) and \( \hat{n} \) are integer multiples of \( 2M \) or not.
1. \( \hat{n}, \hat{n} \notin 2M \): Eqn. 21 is satisfied with no conditions on the prototype filters.

2. \( \hat{n} = 2Ml, \hat{n} \notin 2M \): Since \( \hat{n} \) is even Eqn. 21 is equivalent to \( \frac{1}{2}M(\zeta + \eta) = \delta(2Ml) \). Equivalently:

\[
\sum_{n} h(Mn + n_1)\theta(\zeta - M - n_1 + 2Ml - \frac{\alpha}{2}) = \frac{2}{M} \delta(l).
\]  

(24)

3. \( \hat{n} \notin 2M, \hat{n} = 2Ml \): Since \( \hat{n} \) is even Eqn. 21 is equivalent to \( \frac{1}{2}M(\zeta - \eta) = 0 \). Equivalently:

\[
\sum_{n} (1)^n h(Mn + n_1)\theta(\zeta - M + n_1 - \alpha - 2Ml) = 0
\]  

(25)

4. \( \hat{n} = 2Ml, \hat{n} = 2Mk \): This can happen only if \( \alpha \) is an even integer. In this case, Eqn. 21 is equivalent to \( \frac{1}{2}M(\zeta - \eta) = \delta(2Ml) \). That is

\[
h(2Mn + \frac{\alpha}{2})\theta(\zeta - M - n_1 + 2Ml - \frac{\alpha}{2}) = \frac{1}{M} \delta(l)
\]  

(26)

Eqn. 24 and Eqn. 25 can be compactly written as Eqn. 8. Moreover Eqn. 26 becomes \( P_{\delta, \omega}Q_{\delta, \omega} = \frac{2}{\omega} \). That \( P_{\delta, \omega} \) and \( Q_{\delta, \omega} \) are left unspecified by the PR conditions should not come as a surprise since for these indices the modulation vector is zero.

1.2. Class A - DCT/DST II based

In this case \( \alpha \) is an odd integer. Let \( \zeta, \eta, \hat{n} \) and \( \hat{n} \) be as before. Again for PR we require Eqn. 21 with

\[
a(n) = \sum_{l=0}^{n-1} k_l \cos(\pi sn) = \begin{cases} M - \frac{1}{2} & n \in 2M \\ -\frac{1}{2}(-1)^n & \text{otherwise} \end{cases}
\]  

(27)

\[
b(n) = \sum_{l=1}^{M} k_l \cos(\pi sn) = \begin{cases} M - \frac{1}{2} & n \in 2M \\ -\frac{1}{2} & \text{otherwise} \end{cases}
\]  

(28)

As before there are four cases to consider depending on whether \( n \) and \( \hat{n} \) are integer multiples of \( 2Ml \) or not.

1. \( n, \hat{n} \notin 2M \): Eqn. 21 is satisfied with no conditions on the prototype filters.

2. \( n = 2Ml, \hat{n} \notin 2M \): Since \( n \) is even Eqn. 21 is equivalent to \( \frac{1}{2}M(\zeta + \eta) = \delta(2Ml) \). Equivalently:

\[
\sum_{n} h(Mn + n_1)\theta(\zeta - M - n_1 + 2Ml - \frac{\alpha}{2}) = \frac{2}{M} \delta(l).
\]  

(29)

3. \( n \notin 2M, \hat{n} = 2Ml \): Since \( n \) is even Eqn. 21 is equivalent to \( \frac{1}{2}M(\zeta - \eta) = 0 \). Equivalently:

\[
\sum_{n} (1)^n h(Mn + n_1)\theta(\zeta - M + n_1 - \alpha - 2Ml) = 0
\]  

(30)

4. \( n = 2Ml, \hat{n} = 2Mk \): This cannot happen since \( \alpha \) is an odd integer.

Eqn. 8 hence follows.

1.3. Class B - DCT/DST III/IV based

It’s folklore [2]!

B PROOFS FOR MWTFS

2.1. Class A. \( \alpha \) is even. Type 1 (\( M \) is odd)

Want \( \sum_{n} h_0(n) = \sqrt{2M} \). From Eqn. 4a and Eqn. 15

\[
\sqrt{2M} = \frac{\sqrt{2}}{\sqrt{2^M}} + \frac{1}{\sqrt{M}} \sum_{l=0}^{J-1} 2\sqrt{2} \sin(\Theta_l + \frac{\pi}{4})
\]  

(31)

This holds iff \( \sum_{l=0}^{J-1} \sin(\Theta_l + \frac{\pi}{4}) = \frac{M - 1}{2} = J \). That is iff \( \Theta_l = \{\frac{\pi}{4}\}_{l=0} \) for all \( l \in \mathbb{R}(J) \).

2.2. Class A. \( \alpha \) is even. Type 2 (\( M \) is even)

\[
\sqrt{2M} = \frac{\sqrt{2}}{\sqrt{2^M}} + \frac{1}{\sqrt{M}} \sum_{l=0}^{J-1} 2\sqrt{2} \sin(\Theta_l + \frac{\pi}{4})
\]  

(32)

This holds iff \( \sum_{l=0}^{J-1} \sin(\Theta_l + \frac{\pi}{4}) = \frac{M - 1 - 1/\sqrt{2}}{2} > J \), an impossibility. Therefore, there are no Type 2, Class A DCT/DST I based MFBs.

2.3. Class A. \( \alpha \) is odd. Type 1 (\( M \) is even)

\[
\sqrt{2M} = \frac{1}{\sqrt{M}} + \sum_{l=0}^{J-1} 2\sqrt{2} \sin(\Theta_l + \frac{\pi}{4})
\]  

(33)

This holds iff \( \sum_{l=0}^{J-1} \sin(\Theta_l + \frac{\pi}{4}) = \frac{M}{2} = J \). That is iff \( \Theta_l = \{\frac{\pi}{4}\}_{l=0} \) for all \( l \in \mathbb{R}(J) \).

2.4. Class A. \( \alpha \) is odd. Type 2 (\( M \) is odd)

\[
\sqrt{2M} = \frac{1}{\sqrt{M}} + \sum_{l=0}^{J-1} 2\sqrt{2} \sin(\Theta_l + \frac{\pi}{4})
\]  

(34)

This holds iff \( \sum_{l=0}^{J-1} \sin(\Theta_l + \frac{\pi}{4}) = \frac{M - 1/\sqrt{2}}{2} > J \), an impossibility. Therefore, there are no Type 2, Class A DCT/DST I based MFBs.

REFERENCES


