On Cosine-Modulated Wavelet Orthonormal Bases

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Abstract—Recently, multiplicity $M$, $K$-regular, orthonormal wavelet bases (that have implications in transform coding applications) have been constructed by several authors. This paper describes and parameterizes the cosine-modulated class of multiplicity $M$ wavelet tight frames (WTF's). In these WTF's, the scaling function uniquely determines the wavelets. This is in contrast to the general multiplicity $M$ case, where one has to, for any given application, design the scaling function and the wavelets. Several design techniques for the design of $K$ regular cosine-modulated WTF's are described and their relative merits discussed. Wavelets in $K$-regular WTO's may or may not be smooth. Since coding applications use WTO's with short length scaling and wavelet vectors (since long filters produce ringing artifacts, which is undesirable in, say, image coding), many smooth designs of $K$ regular WTO's of short lengths are presented. In some cases, analytical formulas for the scaling and wavelet vectors are also given. In many applications, smoothness of the wavelets is more important than $K$ regularity. We define smoothness of filter banks and WTO's using the concept of total variation and give several useful designs based on this smoothness criterion. Optimal design of cosine-modulated WTO's for signal representation is also described. All WTO's constructed in this paper are orthonormal bases, and we believe that this is always the case.

I. INTRODUCTION

RECENTLY, orthonormal bases of compactly supported wavelets have received considerable attention in the signal processing community, both as a tool for signal analysis [5]–[8] and signal representation [9]–[14]. Several authors have tried to use wavelets for image compression [10]–[15]. It is well known that image coding using wavelets is a special case of subband coding using particular sets of filters called scaling and wavelet vectors. However, this specialization, and the mathematical theory of wavelets that comes with it, gives new results and insights into the performance of image compression methods using subband coding [10].

Transform coding (using block transforms, subband coders, or lapped transforms) is known to be an effective technique for the representation of speech and image signals at medium to low bit rates [16]. Although subband coders solve the blocking effects associated with block transforms (like the DCT), they come with added cost in design and implementation. Subband coders using unitary filter banks (which are also known as lapped transforms) partially alleviate this problem of design and implementation. Moreover, they are precisely the class of filter banks associated with orthonormal wavelet bases. More recently, a new class of unitary filter banks called cosine-modulated filter banks (which are also known as extended lapped transforms) have been developed. These filter banks are easy to design and implement.

In her landmark paper [4], Daubechies constructs compactly supported orthonormal wavelet bases by constructing a special sequence: the scaling vector $h_0$ that satisfies a set of linear and quadratic conditions. The quadratic conditions ensure that $h_0$ is a filter in a two-channel unitary filter bank. The linear conditions (one of which is necessary for the existence of wavelets) ensure that the Fourier transform of $h_0$ is sufficiently flat at the origin. These linear conditions are also known as regularity conditions because for the wavelet to be $K-1$ times differentiable, it is necessary that $h_0$ is $K$-regular (i.e., $K-1$ derivatives of $H_0(\omega)$ vanish at the origin). However, regularity of $h_0$ does not guarantee any degree of differentiability (or smoothness) of the wavelet. Another important consequence of regularity is that the zeroth through $K-1$ moments of the wavelet are zero.

Using the well-known theory of $M$-channel unitary filter banks, the authors and several others have recently constructed orthonormal bases (or tight frames) of multiplicity $M$, $K$-regular, compactly supported wavelets [1]–[3]. Multiplicity $M$ wavelet bases give a more flexible time-frequency resolution tradeoff than octave-band multiplicity 2 wavelet bases [3]. One difference between the multiplicity 2 case and the multiplicity $M$ case is that in the former, once the scaling vector and, hence, the scaling function is determined, the wavelet vector $h_1$ and the corresponding wavelet is precisely determined modulo translation. However, in the multiplicity $M$ case, there are $\binom{M-1}{2}$ free parameters that determine the $M-1$ wavelet vectors $h_i$ (and hence the corresponding wavelets) once the scaling vector is given.

Since there is a rich family of compactly supported WTF's, one can pick a tight frame that is suited to any particular application. This involves choosing the scaling function and then the wavelets. In the wavelet literature, there has been some recent work on the choice of “optimal” and “robust” scaling function or, equivalently, the scaling vector [11]–[13]. The scaling vector is generically of length $N = MK$ and determined by $(M-1)(K-1)$ parameters. All properties of the multiresolution analysis is determined by the scaling function. However, when $M > 2$, to get a WTF, one has to design the wavelets as well. For large $M$, the design of the WTF becomes difficult because of the enormous number of free parameters $\binom{M-1}{2}$.

In this paper, we solve this problem by constructing a class of multiresolution analyses where the scaling function uniquely determines the wavelets [17]. We obtain this by using
the recently developed theory of unitary cosine-modulated filter banks [18]–[21]. We explicitly parameterize all compactly supported multiplicity \( M \) cosine-modulated WTF’s. There are a number of advantages to this approach:

1) First, the Fourier transform of the wavelets and scaling functions occupy adjacent bands in the spectrum, achieving the primary goal of flexible time-frequency resolution, which is the very reason for considering multiplicity \( M \) WTF’s.

2) Second, the design is simple. Although a general WTF is determined by \( \binom{M-1}{2} + (M-1)(K-1) \) parameters, where \( N = MK \), a cosine-modulated WTF is determined by just \( N/4 \) parameters. For large \( M \) or \( N \), the savings is remarkable. For example, if one wants to design a multiplicity 64 WTF, the cosine-modulated approach seems to be the only viable one.

3) Analysis/synthesis computations in these wavelet bases can be done very efficiently using a combination of two-channel unitary filter banks and the DCT.

In many applications, in numerical analysis and approximation theory using wavelets, it is desirable to have first \( K - 1 \) moments of the wavelets vanish [22], [23], [14]. This also means that the scaling function and its translates can approximate polynomials of degree \( K - 1 \) exactly. In the multiplicity \( M \) case, one can explicitly give formulas for the autocorrelation of \( K \) regular scaling vectors [3]. Such an explicit characterization of \( K \)-regular cosine-modulated scaling vectors does not seem to be feasible. However, using the parameterization developed in this paper, one can design \( K \) regular scaling vectors and, hence, WTF’s. In general, one has to solve a set of nonlinear equations, and hence, \( K \)-regular WTF’s are designed numerically. However, for small \( K \) and \( M \), they may be designed analytically. Our analytical examples illustrate the important fact that regularity of the scaling vector may not imply regularity of the wavelets in the WTF. In addition, for fixed \( N \) and \( K \), there may be multiple solutions (that are not distinct spectral factors of a fixed autocorrelation function) to the nonlinear equations. However, for small \( K \) and \( M \), we give smooth WTF designs using this approach. For moderate \( M \) and \( K \), the length of the scaling vector grows rapidly, making the numerical design difficult. To cope with this problem, we introduce a novel technique (the moments minimization method), which produces smooth designs more consistently than the earlier approach.

In most signal processing applications, the regularity of the scaling vector (or the moment vanishing property of the wavelets) is not important. However, the smoothness of the scaling function and wavelets (i.e., their being "visually" nonfractal) is important. For example, in image processing applications, tree-structured hierarchical lapped transforms have been used for some time. If the tree is deep, then the low-pass filter iterates, and the scaling function samples can be identified. Said differently, in such applications, iterates of the scaling vector must not look bad. We propose two new techniques for the design of such "smooth" scaling vectors without using the usual regularity approach. The first is based on the intuitive idea that if one were to design a scaling function that optimally represents an artificial smooth signal (like a sinusoid, for example), then the scaling function would tend to be smooth. Using the theory of design of optimal multiresolution analyses in [12], we show how smooth cosine-modulated WTF’s can be designed. The second approach, which is based on a new definition of smoothness, seems to be well suited for designing smooth WTF’s. This approach also is shown to lead to better filter bank designs.

All explicit examples of WTF’s constructed in this paper are orthonormal bases. We conjecture that all cosine-modulated WTF’s are orthonormal bases.

II. OVERVIEW OF RELEVANT RESULTS

A. Wavelet Frames and Filter Banks

Fig. 1 shows an \( M \)-channel filter bank with analysis filters \( h_n \) and synthesis filters \( g_n \).

The filter bank is a perfect reconstruction (PR) filter bank if \( y(n) = x(n) \). Under certain conditions, PR filter banks are associated with wavelet frames for \( L^2(\mathbb{R}) \) [1], [24]. This association is a correspondence between the filters and the scaling and wavelet vectors associated with the wavelet frame (see Table 1).

PR filter banks, where the analysis and synthesis filters satisfy \( g_n(n) = h_n(-n) \), are called unitary filter banks. Unitarity of a filter bank is equivalent to the following quadratic
conditions on the filters:
\[ \sum_k h_i(k)h_j(k + ML) = \delta(l)\delta(i - j). \] (1)

The term unitary comes from the fact that the polyphase component matrix of the filter bank is unitary on the unit circle [25]. Unitary filter banks are also known as paraunitary filter banks [25].

A unitary filter bank where the low-pass filter satisfies the additional linear constraint
\[ \sum_{k=0}^{N-1} h_0(k) = \sqrt{M} \] (2)
gives rise to a wavelet tight frame [1]. This filter is the unitary scaling vector, and the remaining filters in the filter bank are the unitary wavelet vectors. The scaling and wavelet vectors determine the scaling function and wavelets defined by the following equations:
\[ \psi_i(t) = \sqrt{M} \sum_k h_i(k) \psi_0(Mt - k) \quad i = 0, \ldots, M - 1. \] (3)
The \((M-1)\) wavelets \(\psi_i(t), i = 1, \ldots, M - 1,\) their translates, and dilates by powers of \(M\) form a wavelet tight frame. That is, any \(f(t) \in L^2(\mathbb{R})\) can be represented in the form
\[ f(t) = \sum_{i=1}^{M-1} \sum_{j,k} <f, \psi_{i,j,k}(t) > \psi_{i,j,k} \]
where \(\psi_{i,j,k}(t) = M^{j/2}\psi_i(M^{j}t - k).\)

It is well known in the multiplicity 2 case that the smoothness (degree of differentiability) of the scaling function is usually hard to determine [4], [26], [27]. The situation is the same in the multiplicity \(M\) case as well. However, a necessary condition for \(K-1\) derivatives of \(\psi_0(t)\) (and hence the wavelets) to exist is that the zeroth through \(K-1\) moments of the wavelets vanish. This is equivalent to the Fourier transform of the scaling vector being \(K\)th order flat at the origin. Scaling vectors with this property are called \(K\)-regular scaling vectors. More formally, we have the following definition.

**Definition 1:** A scaling vector is said to be \(K\) regular if its \(Z\) transform is of the form
\[ H_0(z) = (1 + z^{-1} + \cdots + z^{-(M-1)})K P(z) \] (4)
for maximal possible \(K\), and \(P(z)\) is a polynomial in \(z^{-1}\).

It can be shown that the minimal length \(K\)-regular scaling vectors are generically of length \(N = MK\) [3]. For example, the smallest length scaling vector is the Haar scaling vector, which is given by
\[ h_0(n) = \begin{cases} \frac{1}{\sqrt{M}} & \text{for } n \in 0, 1, \ldots, M - 1 \\ 0 & \text{otherwise} \end{cases} \] (5)

Clearly, this scaling vector is \(1\)-regular. From (2) and (1), it can be shown that every scaling vector is necessarily \(1\)-regular. For all multiplicities \(M, \) \(K\)-regular scaling vectors can be constructed [3], and the corresponding WTF’s are called \(K\)-regular WTF’s.

**B. Cosine-Modulated FIR Filter Banks**

Cosine-modulated FIR filter banks are a special class of unitary FIR filter banks, where the analysis filters \(h_i(n)\) are all cosine-modulated of a low-pass linear-phase prototype filter \(g(n)\) [18]. The fundamental idea behind cosine-modulated filter banks is the following: In an \(M\)-channel filter bank, the analysis and synthesis filters are meant to approximate ideal \(M\)th band filters, which are shown in Fig. 2. The passbands of these filters occupy adjacent frequency channels that are \(\pi/M\) apart. Given a real, prototype filter \(g(n)\) with passband \([\frac{-\pi}{2M}, \frac{\pi}{2M}]\), if we modulate it by \(\cos ((2i + 1)\pi/M n + \epsilon_i)\), where \(\epsilon_i\) is an arbitrary phase, the passband of the resultant filter occupies the desired bands in Fig. 2. Can an FIR prototype filter be chosen so that the resultant filters form a perfect reconstruction filter bank? This question has been answered in the affirmative independently by Malvar [19] and Koilpillai [18], [25], [20], [21]. We will use the complete parametrization of cosine-modulated filter banks of length \(N = 2Mm\) in [18] to obtain our explicit parametrization of cosine-modulated WTF’s.

In this paper, we will use the modulation in [18]. That is
\[ h_i(n) = c_{i,n} g(n) \] (6)
where \(g(n)\) is an even-symmetric prototype filter of length \(N = 2Mm\) for some nonnegative integer \(m\) and
\[ c_{i,n} = \cos \left( \frac{\pi}{2M} (2i + 1)(n - \frac{N - 1}{2}) + \tau_i \right). \] (7)
The phase factor \(\tau_i\) can be taken to be \((-1)^i \pi/2\). In order to see how unitarity of the cosine-modulated filter bank is reflected on the prototype filter, it is convenient to define the polyphase components of the prototype filter.
\[ G(z) = \sum_{n=0}^{2M-1} \left\{ \sum_{i=0}^{m-1} g(2Mi + n) z^{-2Mi} \right\} z^{-n} \]
\[ = \sum_{n=0}^{2M-1} z^{-n} G_n(z^{2M}). \] (8)

Each of \(G_n(z)\) is an FIR sequence of length \(m\), and there are \(2M\) of them accounting for the length \(2Mm\) of \(g(n)\). A cosine-modulated filter bank is unitary iff for \(n = 0, \ldots, M - 1\) [18]
\[ G_n(z^{-1}) G_n(z) + G_{M+n}(z^{-1}) G_{M+n}(z) = \frac{2}{M}. \] (9)
From the even symmetry of \(G(z)\), it follows that
\[ G_n(z) = z^{-(m-1)} G_{2M-1-n}(z^{-1}). \] (10)
Therefore, if we define \(J = \left\lceil \frac{M}{2} \right\rceil\), for even \(M\), (9) need hold only for \(n = 0, \ldots, J - 1\). For odd \(M\), however, one additionally requires
\[ G_J(z) G_J(z^{-1}) = \frac{1}{M}. \] (11)
The most general FIR solution to (11) is \( G_J(z) = \pm \sqrt{\frac{1}{M}} z^{-k} \) for some integer \( k \). As for (9), each of the \( J \) pairs of filters \( G_n(z), G_{M+n}(z) \) are characterized by \( m \) angle parameters \( \theta_{n,l} \), \( l = 0, \ldots, m-1 \) [18]. Indeed, one has the following lattice factorization of these pairs:

\[
\begin{bmatrix}
G_n(z) \\
G_{n+M}(z)
\end{bmatrix} = \sqrt{\frac{2}{M}} \prod_{l=0}^{m-1} \begin{bmatrix}
\cos(\theta_{n,l}) & \sin(\theta_{n,l}) \\
\sin(\theta_{n,l}) & -\cos(\theta_{n,l})
\end{bmatrix} \begin{bmatrix}
\cos(\theta_{n,0}) \\
\sin(\theta_{n,0})
\end{bmatrix}.
\]

(12)

Therefore, a unitary filter bank is parameterized by \( Jm = \lceil \frac{M}{2} \rceil \approx N/4 \) parameters \( \theta_{n,l} \).

III. COSINE-MODULATED WAVELET TIGHT FRAMES

The wavelet tight frames theorem states that an FIR unitary filter bank gives rise to a WTF if the linear constraint in (2) (or equivalently \( H_0(1) = 1 \)) holds. Since the cosine-modulated filter bank is expressed in terms of the polyphase components of the prototype filter, and the linear constraint is directly on the low-pass filter, it is convenient to express the filters in terms of the polyphase components. The modulation vector satisfies, \( c_{n+2M} = (1)^n c_n \), and therefore

\[
H_i(z) = \sum_{n=0}^{2M-1} c_{i,n} z^{-n} \left[ \sum_{l=0}^{m-1} (-1)^l g(2Ml+n) z^{-2Ml} \right] = \sum_{n=0}^{2M-1} c_{i,n} z^{-n} G_n(-z^{-2M}).
\]

(13)

For \( H_0(1) \) to be equal to one, we therefore require from (13)

\[
\sum_{n=0}^{2M-1} c_{0,n} G_n(-1) = \sqrt{M}.
\]

(14)

The polyphase components \( G_n(z) \) have to merely satisfy the \textit{single} constraint above. An interesting result that occurs is that this single constraint decouples into \( J \) constraints over each polyphase component pair. Moreover, despite the nonlinear dependence of a polyphase pair on the lattice angles generating it, the constraint reduces to a simple linear constraint on the angles. In addition, it turns out that while one has to consider the case of odd and even \( M \) separately for cosine-modulated filter banks, the WTF conditions on each lattice is one clean formula that is independent of the parity of \( M \).

We first evaluate \( G_n(-1) \) in terms of the lattice angle parameters. For \( n \in 0, 1, \ldots, M-1 \), it is easy to see that \( G_n(-1), G_{M+n}(-1), G_{M-1-n}(-1), \) and \( G_{2M-1-n}(-1) \) depend on the same set of angle parameters. For all \( M \) (12) implies

\[
\begin{bmatrix}
G_n(-1) \\
G_{n+M}(-1)
\end{bmatrix} = \sqrt{\frac{2}{M}} \prod_{l=0}^{m-1} \begin{bmatrix}
\cos(\theta_{n,l}) & -\sin(\theta_{n,l}) \\
\sin(\theta_{n,l}) & \cos(\theta_{n,l})
\end{bmatrix} \begin{bmatrix}
\cos(\theta_{n,0}) \\
\sin(\theta_{n,0})
\end{bmatrix} = \sqrt{\frac{2}{M}} \begin{bmatrix}
\cos(\theta_{n,0}) \\
\sin(\theta_{n,0})
\end{bmatrix}.
\]

(15)

In addition, from (10)

\[
\begin{bmatrix}
G_{2M-1-n}(-1) \\
G_{M-1-n}(-1)
\end{bmatrix} = \begin{bmatrix}
(-1)^{(m-1)} G_{n}(-1) \\
(-1)^{(m-1)} G_{M+n}(-1)
\end{bmatrix} = \sqrt{\frac{2}{M}} \begin{bmatrix}
(-1)^{(m-1)} \cos(\theta_{n}) \\
(-1)^{(m-1)} \sin(\theta_{n})
\end{bmatrix}.
\]

(16)

Together, we have

\[
\begin{bmatrix}
G_{n}(-1) \\
G_{M+n}(-1)
\end{bmatrix} = \sqrt{\frac{2}{M}} \begin{bmatrix}
\cos(\theta_{n}) \\
\sin(\theta_{n})
\end{bmatrix}.
\]

(17)

For odd \( M \), \( G_J(z) = \pm \sqrt{\frac{1}{M}} z^{-k} \), and from the even symmetry of \( g(n) \), \( G_{M+J}(z) = \pm \sqrt{\frac{1}{M}} z^{-(m-1-k)} \). Therefore

\[
\begin{bmatrix}
G_{J}(-1) \\
G_{M+J}(-1)
\end{bmatrix} = \pm \sqrt{\frac{1}{M}} \begin{bmatrix}
(-1)^{k} \\
(-1)^{(m-1-k)}
\end{bmatrix}.
\]

(18)

Finally, if we define \( \alpha_n = \frac{\pi}{2M} (n - Mm + \frac{1}{2}) + \frac{\pi}{4} \), it is easy to verify that

\[
\begin{bmatrix}
c_{0,n} \\
c_{0,n+M} \\
c_{0,M-1-n} \\
c_{0,2M-1-n}
\end{bmatrix} = \begin{bmatrix}
\cos(\alpha_n) \\
-\sin(\alpha_n) \\
(-1)^{(m-1)} \cos(\alpha_n) \\
(-1)^{(m-1)} \sin(\alpha_n)
\end{bmatrix}.
\]

(19)

In addition, for odd \( M \)

\[
\begin{bmatrix}
c_{0,J} \\
c_{0,J+M}
\end{bmatrix} = \begin{bmatrix}
\cos(\frac{\pi}{2} (1 - m)) \\
-\sin(\frac{\pi}{2} (1 - m))
\end{bmatrix}.
\]

(20)

We consider the case of even and odd \( M \) separately. When \( M \) is even, the one has a cosine-modulated WTF iff we have (21), which appears at the bottom of the page, or equivalently

\[
\sum_{n=0}^{J-1} \sin(\frac{\pi}{4} + \alpha_n + \Theta_n) = \frac{M}{2} = J.
\]

(22)
Clearly, each of the \( J \) sinusoids on the left-hand side of the equation has to be one, and therefore

\[
\Theta_n = \frac{\pi}{2} m - \frac{\pi}{4M} (2n + 1). \tag{23}
\]

For odd \( M \), there are two additional terms in the expression

\[
\sum_{n=0}^{2M-1} G_n(-1)c_{0,n} \text{ corresponding to } n = J \text{ and } n = J + M.
\]

In this case, one has a WTF iff

\[
\sqrt{M} = \sqrt{\frac{2}{M}} \left\{ \sum_{n=0}^{M-1} \cos(\alpha_n + \Theta_n) + \sin(\alpha_n + \Theta_n) \right\} + c_{0,J}G_J(-1) + c_{0,M+J}G_{M+J}(-1).
\]

For fixed \( m \), since \( c_{0,J} \) or \( c_{0,M+J} \) is zero and since \( G_J(-1) \) and \( G_{M+J}(-1) \) are of the form \( \pm \sqrt{\frac{1}{M}} \)

\[
\sum_{n=0}^{J-1} \sin\left(\frac{\pi}{4} + \alpha_n + \Theta_n\right) = \frac{M-1}{2}.
\]

Clearly, there is no solution if the right-hand side is \( J + 1 \). This fixes the sign of \( G_J(z) \) and \( G_{J+M}(z) \). Moreover, in this case, since each of the sinusoids on the left-hand side is one, (23) holds once again. We have therefore proved the following theorem.

**Theorem 1**: Cosine-modulated multiplicity \( M \) WTF’s exist for all \( M \). All such WTF’s with scaling function and wavelets supported in \( [0, \frac{2M(m-1)}{M-1}] \) are parameterized by \( J(m-1) \) free parameters or \( Jn \) angles \( \theta_{n,i} \), \( n = 0, \ldots, J-1 \), and \( i = 0, \ldots, m-1 \) such that \( \sum_{n=0}^{J-1} \theta_{n,i} = \frac{\pi}{2} m - \frac{\pi}{4M} (2n + 1) \).

**Remark**: The WTF was imposed by imposing one condition, viz., (2) on the cosine-modulated filter bank that has \( Jn \) parameters. Therefore, one expects the WTF to have \( Jn - 1 \) parameters and not \( J(m-1) \), as Theorem 1 shows. The linear constraint (on \( h_0 \)) interestingly becomes a linear constraint on the angles of each of the \( J \) lattices (recall that the angles are related to \( h_0 \) nonlinearly). For odd \( M \), one additional has the sign of the delays \( G_J(z) \) and \( G_{J+M}(z) \) constrained such that \( G_J(-1)c_{0,J} + G_{M+J}(-1)c_{0,M+J} = \sqrt{\frac{1}{M}} \). Why does this happen? From the quadratic condition (1), by taking the \( Z \) transform on both sides, we get

\[
1 = Z(\sum_k h_0(k)h_0(k+M)) = [M]H_0(z)H_0(z^{-1})
\]

and therefore

\[
\frac{1}{M} \sum_{k=0}^{M-1} |H_0(e^{i(\omega + \frac{2\pi k}{M})})|^2 = 1.
\]

Since \( H_0(e^{i\theta}) = \sqrt{M} \), we have for \( k = 1, \ldots, M-1 \)

\[
H_0\left(\frac{2\pi k}{M}\right) = 0.
\]

From the symmetry of the roots of unity, the \( (M-1) \) complex constraints reduce to \( M-1 \) real constraints. Together with (2), there are \( M \) linear constraints, and with linear phase, this reduces to \( M/2 = J \) constraints.

Two examples of cosine-modulated WTF’s, one each for odd and even \( M \), respectively, follow.

**Example 1**: \( M = 2 \) and \( N = 4 \): Since \( N = 2M \), there are no free parameters. Since \( J = 1 \), there is precisely one lattice with one angle parameter, which is given from (23) by

\[
\Theta_0 = \theta_{0,0} = \frac{\pi}{2} m - \frac{\pi}{4M} (2n + 1) = \frac{\pi}{2} - \frac{\pi}{8} = \frac{3\pi}{8}.
\]

The prototype filter is given by

\[
g = \sqrt{\frac{2}{M}} \left[ \cos(\theta_{0,0}) \sin(\theta_{0,0}) \sin(\theta_{0,0}) \cos(\theta_{0,0}) \right].
\]

The corresponding scaling and wavelet vectors (from (6)) are given by

\[
h_0 = \left[ \begin{array}{c} \frac{1}{\sqrt{2}} \sqrt{2} \frac{1}{\sqrt{4}} \frac{1}{\sqrt{2}} \frac{1}{2} \sqrt{2} \end{array} \right] \quad \text{and} \quad h_1(n) = \left[ -\frac{1}{\sqrt{2}} \frac{1}{2} \frac{1}{\sqrt{2}} \frac{1}{2} \right].
\]

The scaling function, wavelet, their Fourier transforms and the discrete-time Fourier transform of the corresponding filters are shown in Fig. 4. The scaling function and wavelet (both of which are supported in \( [0,3] \)) exhibit “kinks” in their graph, whereas their Fourier transforms are well behaved. These functions are not as smooth as the corresponding length 4 example in [4] (where the WTF is not cosine modulated).

**Example 2**: \( M = 5 \) and \( N = 10 \): Again, since \( N = 2M \), there are no free parameters. In this case, since \( J = \lceil \frac{M}{2} \rceil = 2 \), there are two lattices, each with one angle parameter given by (from (23))

\[
\Theta_0 = \theta_{0,0} = \frac{\pi}{2} m - \frac{\pi}{4M} (2n + 1) = \frac{\pi}{2} - \frac{9\pi}{20} = \frac{9\pi}{20}
\]

\[
\Theta_1 = \theta_{0,0} = \frac{\pi}{2} m - \frac{\pi}{4M} (2n + 1) = \frac{3\pi}{2} - \frac{7\pi}{20} = \frac{7\pi}{20}.
\]

Since \( M \) is odd, \( G_J(z) \) is an arbitrary delay. We will choose \( g(J) = g(J + M) = \frac{1}{\sqrt{2}} \). Although, for a cosine-modulated filter bank, it is irrelevant whether these two numbers are \( \frac{\sqrt{2}}{2} \) or \( -\frac{\sqrt{2}}{2} \), it is necessary that they be both positive for a WTF. The prototype filter \( g \) is given by the expression at the bottom of the next page. The scaling and wavelet vectors are readily obtained. (In Section III-A, we give formulas for them.) The scaling function, wavelets, their Fourier transforms and the Fourier transforms of the corresponding filters are shown in Fig. 6.

### A. Explicit Formulae for WTOs with \( N = 2M \)

If \( N = 2M \) (as in the two cases above), the cosine-modulated WTF is unique since there are no free parameters. Therefore, an explicit formula for the scaling and wavelet vectors can be obtained. This has important consequences in transform coding applications (especially image coding), where very long filters would exhibit ringing artifacts, and one has to keep \( N \) as small as possible while simultaneously having \( M \) large [21].
When \( N = 2M \) (and \( M \) even), it is easy to see that the prototype filter is of the form

\[
\begin{bmatrix}
g(0) \\
g(J - 1) \\
g(J) \\
g(M - 1) \\
g(M) \\
\vdots \\
g(2M - 1)
\end{bmatrix} = \sqrt{\frac{2}{M}} \begin{bmatrix}
\cos(\Theta_0) \\
\cos(\Theta_{J-1}) \\
\cos(\Theta_{J-1}) \\
\sin(\Theta_0) \\
\sin(\Theta_{J-1}) \\
\sin(\Theta_{J-1}) \\
\cos(\Theta_0)
\end{bmatrix}.
\] (32)

Therefore, for \( n \in 0, \ldots, J - 1 \), \( g(n) = \cos(\frac{\pi}{2} - (2n + 1)\frac{\pi}{4M}) \). In addition, for \( n \in 0, 1, \ldots, J - 1 \),
\[
g(J + n) = \sin(\frac{\pi}{2} - (2(J - 1 - n) + 1)\frac{\pi}{4M}) = \cos(\frac{\pi}{2} - (2(J + n) + 1)\frac{\pi}{4M}).
\]
This implies that for \( n \in 0, \ldots, M - 1 \), \( g(n) = \cos(\frac{\pi}{2} - (2n + 1)\frac{\pi}{4M}) \).

Similarly, it can be shown that for \( n \in 0, \ldots, M - 1 \),
\[
g(M + n) = \sin(\frac{\pi}{2} - (2n + 1)\frac{\pi}{4M}).
\]
Equivalently, \( g(M + n) = \cos(\frac{\pi}{2} - (2(M + n) + 1)\frac{\pi}{4M}) \).

Putting it all together, we have the following simple formula for the prototype filter:

\[
g(n) = \sin\left(\frac{\pi}{4M}(2n + 1)\right).\] (33)

One can similarly do an analysis for the case when \( M \) is odd and show that the prototype filter is given by (33). Therefore, the scaling and wavelet vectors are given by (34), which appears at the bottom of the next page.

**Theorem 2**: Scaling/wavelet vectors in cosine-modulated WTF’s with \( N = 2M \) are given by (34).

### B. Orthonormality of Cosine-Modulated Wavelet Tight Frames

All examples of cosine-modulated WTF’s that we construct in this paper are verified to be orthonormal based on the characterization of orthonormality of multiplicity \( M \) WTF’s in [3] and the extension of the ideas in [28]. Based on our experiments, we make the following conjecture.

**Conjecture 1**: Every multiplicity \( M \) cosine-modulated wavelet tight frame is necessarily an orthonormal wavelet basis.

### IV. Higher Regularity Cosine-Modulated WTF’s

In many applications, using wavelets, especially in numerical analysis and the solution of partial differential equations,
it is desirable to have as many moments of the wavelets as possible vanish. Usually, this is not required in signal processing. However, when signals have a local polynomial behavior, this property of the wavelets is useful. There is yet another reason that this property may be useful in signal processing: If filter banks are iterated in a tree structure, regularity is known to give reasonable iterated filter responses [4]. This section discusses the design of such regular cosine-modulated WTF’s and some of the difficulties associated with it.

The basic idea is to impose the regularity condition on the parameterization of cosine-modulated WTF’s. By solving the set of nonlinear equations associated with these constraints one obtains regular WTF’s. For a given $K$, there is a smallest $N = 2Mm$ such that the scaling vector can be $K$ regular. Such scaling vectors will be called maximally regular cosine-modulated scaling vectors. The length $N_{\text{min}}$ of maximally regular scaling vectors grows rapidly with $K$. For $K$-regularity (from (4)), we want $k = 1, \ldots, K - 1$

$$\left( \frac{d}{dz} \right)^k H_0(z) = 0 \quad (35)$$

for $z = e^{i2\pi l/M}$, $l = 1, \ldots, M - 1$. From the symmetry of the roots of unity, these complex constraints for each $k$ reduce to a set of $M - 1$ real constraints. Hence, there are $(K - 1)(M - 1)$

$$h_1(n) = g(n)c_{\nu,n} = g(n)\cos \left( \frac{\pi}{2M} (2i + 1)(n - \frac{2M - 1}{2}) + (-1)^i \frac{\pi}{4} \right) = g(n)(-1)^i \sin \left( \frac{\pi}{2M} (2i + 1)(n + \frac{1}{2}) + \frac{\pi}{4} (-1)^i \right)$$

$$= \sqrt{\frac{1}{2M}} \left[ \sin \left( \frac{\pi}{4} \frac{2i + 1}{n + 0.5} - (-1)^i \frac{\pi}{4} \right) - \sin \left( \frac{\pi}{4} \frac{2i + 1}{n + 0.5} - (-1)^i \frac{\pi}{4} \right) \right]. \quad (34)$$
constraints. Since \( J(m - 1) \), the number of parameters has to be at least as many as the number of constraints for the maximally regular case

\[
N_{\text{min}} = 2M \left( \left\lceil \frac{(K-1)(M-1)}{J} \right\rceil + 1 \right).
\]

Fig. 5. Example 3: \( M = 2, K = 2, N = 8 \). Case 1: (a) Scaling function \( \psi_0(t) \); (b) \( \hat{\psi}_0(\omega) \); (c) 20 log \( |H_0(\omega)| \); (d) wavelet \( \psi_1(t) \); (e) \( \hat{\psi}_1(\omega) \); (f) 20 log \( |H_1(\omega)| \) and Case 2; (g) scaling function \( \psi_0(t) \); (h) \( \hat{\psi}_0(\omega) \); (i) 20 log \( |H_0(\omega)| \); (j) wavelet \( \psi_1(t) \); (k) \( \hat{\psi}_1(\omega) \); (l) 20 log \( |H_1(\omega)| \).

Fig. 6. Example 4—\( M = 2, k = 3, N = 12 \): (a) Scaling function \( \psi_0(t) \); (b) \( \hat{\psi}_0(\omega) \); (c) 20 log \( |H_0(\omega)| \); (d) wavelet \( \psi_1(t) \); (e) \( \hat{\psi}_1(\omega) \); (f) 20 log \( |H_1(\omega)| \).

This is in contrast to the case of general (not cosine-modulated case) WTE case, where maximally regular scaling vectors are of length \( N = MK \) [3].

For design purposes, it is convenient to write (34) in a form that makes the \( (K-1)(M-1) \) constraints on \( h_0 \) transparent:
For $l \in 1, \ldots, M - 1$ and $k \in 1, \ldots, K - 1$, we have [3]

$$\eta_{k,l} \defeq \sum_n (Mn)^k h_0(Mn) = \sum_n (Mn + l)^k h_0(Mn + l) \defeq \eta_{k,l}. \quad (37)$$

We now discuss a simple example of regular cosine-modulated WTF design.

**Example 3:** Let $M = 2$ and $K = 2$, in which case, $N_{\text{min}} = 8$. There is one free parameter and one the following nonlinear equation (which can be solved analytically)

$$\sum_n h_0(2n + 1) = \sum_n h_0(n). \quad (38)$$

First, from (23)

$$\Theta_0 = \theta_{0,0} + \theta_{0,1} = \frac{7\pi}{8}. \quad (39)$$

It can be easily shown (after straightforward algebraic manipulations) that (38) in conjunction with (39) is equivalent to

$$\sin(2\theta_{0,1}) - \cos(2\theta_{0,1}) = (1 + \sin(\frac{\pi}{4}))/2.$$

**Table II**

<table>
<thead>
<tr>
<th>$h_0(n)$</th>
<th>$h_0(n)$</th>
<th>$h_0(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.129465111606789</td>
<td>0.31034225505271</td>
<td>0.01169999987716</td>
</tr>
<tr>
<td>0.0293456532369909</td>
<td>0.42974996707393</td>
<td>0.03055037515103</td>
</tr>
<tr>
<td>0.27513617647985</td>
<td>0.11199220979935</td>
<td>0.41909124944818</td>
</tr>
<tr>
<td>0.342411165690656</td>
<td>-0.22908827165584</td>
<td>0.32350728844767</td>
</tr>
<tr>
<td>0.29359057475641</td>
<td>0.542411165690656</td>
<td>0.34153593871312</td>
</tr>
<tr>
<td>0.11191862079935</td>
<td>0.27019617647985</td>
<td>0.18997772555776</td>
</tr>
<tr>
<td>0.60374956707222</td>
<td>-0.25835330329608</td>
<td>-0.9566078983163</td>
</tr>
<tr>
<td>0.31034225505271</td>
<td>-0.129465111606789</td>
<td>0.20924612714661</td>
</tr>
</tbody>
</table>

Therefore, there are two solutions to $\theta_{0,1}$ given by

$$\theta_{0,1} = \frac{1}{2} \left[ \frac{\pi}{4} + \arcsin\left(\frac{\sqrt{2} + 1}{4}\right) \right]$$

or

$$\theta_{0,1} = \frac{1}{2} \left[ \frac{\pi}{4} + \pi - \arcsin\left(\frac{\sqrt{2} + 1}{4}\right) \right].$$

where arcsin is assumed to return a value in $[-\pi/2, \pi/2]$. Scaling and wavelet vectors corresponding to the two cases are given in Table II.

Fig. 7 shows that scaling function and wavelet in Case 2 are smoother than the corresponding functions in Case 1. The Fourier transforms also show that the scaling function and
wavelet in Case 1 do not approximate ideal low-pass and band-pass filters, respectively, as well as Case 2. A similar comment applies to the discrete-time Fourier transform of the scaling and wavelet filters.

The example above illustrates a number of features of K-regular cosine-modulated WTF's. First, there may be multiple solutions of maximally regular cosine-modulated scaling vectors (that are not distinct spectral factors, i.e., do not have the same autocorrelation function). This is in contrast to general WTF's, where all maximally regular scaling vectors (see [4] for the multiplicity 2 case and [3] for the multiplicity M case) are necessarily spectral factors and share the same autocorrelation function. Second, that regularity of the scaling vector does not imply "smoothness" the scaling function. The example emphasizes that numerical designs of K-regular cosine-modulated wavelet bases may not be smooth. However, it also suggests that (see Fig. 7(c) and (i)) if one initializes the optimization routine with a good low-pass filter, then there is a reasonable chance that the numerical design would be smooth. A good initial guess for the angles, it turns out, is to set $\theta_{l,m} = \pi/2$ for $l \in 1, \ldots, m - 1$ and all $n$. Another fact that is not so widely appreciated also follows: An increase in the regularity of the scaling vector does not imply a corresponding increase in the smoothness of the scaling function. To illustrate this, we have constructed a 3-regular multiplicity 2 cosine-modulated WTF, the scaling function and wavelets of which are given in Fig. 9 and the corresponding scaling and wavelet vectors given in Table III. Clearly, these functions are not as smooth as the 2-regular example in Example 28 (Case 2).

If the $K$ regularity of the scaling vector (and not the smoothness of the scaling function) is critical, then the method gives a solution. We do not have a proof, but we make the following conjecture.

**Conjecture 2:** There exist multiplicity $M$, $K$-regular wavelet tight frames for all $M$ and $K$.

Since in most numerical analysis applications only low order of $K$ is required, for $M \in 2, \ldots, 8$, 2 regular prototype filters are given in Table IV. Table IV was generated by numerically solving the nonlinear equations in (35). By starting with different initializations, the problem was solved repeatedly and
until "smooth" solutions were found. The prototype filters given correspond to these "smooth" solutions. Moreover, since the prototype filters are even symmetric, \(g(n)\) is only given for \(n \in 0, 1, \ldots, N/2 - 1\). The frequency responses of the filters are shown in Fig. 11.

For \(M = 2\), \(N_{\text{min}} = 2MK\) and, therefore, are \(K - 1\) nonlinear equations for exactly \(K - 1\) unknowns. In addition, for all odd \(M\) \(N_{\text{min}} = 2M(2K - 1)\), and once again, there are \((M - 1)(K - 1)\) equations with precisely \(J(2K - 2) = (M - 1)(K - 1)\) parameters. For these values of \(M\), it is possible to obtain smooth \(K\)-regular solutions. However, for even \(M\) \((M > 2)\), it is difficult to find smooth \(K\)-regular solutions because there are more unknowns than there are equations.

A logical approach is to take care of this situation is to use the extra degrees of freedom to impose some of the equations corresponding to a regularity of \(K + 1\). This approach was found to give very poor solutions. It seems to be the case that imposing some of the \((M - 1)\) equations corresponding to increasing the regularity from \(K\) to \(K + 1\) does more harm than not imposing it at all. We now introduce a technique (the moments minimization method) to solve this problem.

A. Moments Minimization Method

In coding applications, it is desirable to have short filters. If the length is fixed, what is the best that can be achieved in terms of regularity? The moments minimization method provides an answer to this question. Instead of solving (35), this method solves the following equivalent problem (see (37)):

\[
\min_{\theta} \sum_{k=1}^{M-1} \left| \sum_{n=1}^{M-1} |\eta_{k,n} - \eta_{k,0}|^2. \right.
\]

The advantage of this formulation is that it can be solved for an arbitrary length scaling vector (unlike the earlier method of solving nonlinear equations). This method of design in our experiments always gave smooth designs of WTF's. When this method is used to design WTF's with \(N = 4M\) \((M > 3)\), a surprising observation is made. All the solutions were smooth (notice that (35) implies that they could only be 1-regular and, hence, at most differentiable once) and the scaling functions had nearly identical wave shape independent of \(M\). Fig. 8 shows examples of such scaling functions when \(M = 5\) and \(M = 18\), respectively.
TABLE IV

PROTOTYPE FILTERS $g(n)$ FOR $K$-REGULAR COSINE-MODULATED WTF'S

<table>
<thead>
<tr>
<th>$K = 2$</th>
<th>$K = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M = 3, N = 18$</td>
<td>$M = 7, N = 48$</td>
</tr>
<tr>
<td>$M = 5, N = 30$</td>
<td>$M = 9, N = 60$</td>
</tr>
<tr>
<td>$M = 7, N = 42$</td>
<td>$M = 11, N = 72$</td>
</tr>
<tr>
<td>$M = 9, N = 54$</td>
<td>$M = 13, N = 84$</td>
</tr>
</tbody>
</table>

Their remarkable similarity is apparent. Based on numerical experiments, one can obtain a simple approximate analytical formula for all scaling/wavelet vectors of a length $N = 4M$ that is obtained by the moment minimization method. The scaling functions and wavelets of WTF's generated by this formula are always smooth and virtually indistinguishable from those obtained by minimization.

1) Empirical Formula for WTF's with $N = 4M$: The prototype filter in this case is given by

$$g(n) = \frac{1}{2M} \left[ \cos(\beta_n) - \cos \left( \frac{\pi(2n+1)}{4M} \right) \right]$$

for $n = 0, \ldots, 4M - 1$ \hfill (41)

where the $\beta_n$ is given by

$$\beta_n = \begin{cases} \gamma_M + \frac{\pi - \beta_M}{4(M-1)} & \text{for } n = 0, \ldots, M - 1 \\ \beta_M - 1 - n & \text{for } n = M, \ldots, 2M - 1 \\ \beta_M - 2M & \text{for } n = 2M, \ldots, 4M - 1 \end{cases}$$

and $\gamma_M$ is given by the expression at the bottom of the page. The accuracy of this formula may be improved by making the dependence of $\beta_M$ on $n$ quadratic (or even higher order). However, for most practical applications, the above formulas with linear dependence should suffice.

The moments minimization method was introduced to take care of designing approximations to $K$-regular WTF's for all possible lengths $N$. Now, consider the 2-regular example in Table IV corresponding to $M = 4$. The scaling function corresponding to this design is shown in Fig. 9. Consider the same design using the moments minimization method. From (36), $N_{\text{min}} = 24$, and hence, there are $J_{\text{m}} = 2x = 36$ free parameters. Since only three parameters are required for 2-regularity, there is one additional parameter. In the moments minimization method, one minimizes $|\eta_0 - \eta_{-1}|^2 + |\eta_0 - \eta_1|^2 + |\eta_0 - \eta_{-1}|^2 + |\eta_0 - \eta_1|^2$. Notice that the last term could have been replaced by $|\eta_0 - \eta_{-2}|^2$ or $|\eta_0 - \eta_2|^2$. The scaling function for this design is also shown in Fig. 9. The moments minimization method gives a smoother design. Table V shows the difference between partial moments in each case. Although the moments minimization method gives a smoother design, it has smeared the errors among the various moments. The reason is that all the moments are weighted uniformly. One can fix this problem by modifying the method to a weighted moments minimization method:

$$\min \sum_{n=0}^{N-1} w_n |\eta_n - \hat{\eta}_n|^2.$$ \hfill (43)

where $w_n$ is the weight sequence. If, for instance, the $j$th moment must be minimized definitely, then one may choose $w_k$ such that $w_j < w_k, k \neq j$.

V. OPTIMAL AND SMOOTH COSINE-MODULATED WAVELET TIGHT FRAMES

When wavelets are used in the representation of signals, it is more desirable to choose a wavelet that best approximates the signal, rather than a wavelet with a high degree of differentiability (and, hence, regularity of the scaling vector). In this case, the theory of optimal (and robust) representation of signals in WTF's [13], [12] can be applied to the cosine-modulated case. The problem is to design the best multiresolution analysis (i.e., the scaling vector) (for an $a priori$ fixed scale $J$) such that the error of projection of the signal onto scale $J$ has minimum energy. The problem in general is hard to analyze, but under the realistic assumption of the signal being bandlimited, it simplifies considerably.

0.4717 + e^{(-0.00302040424272M^2 + 0.01619797691563M^2 - 0.39479347799199M - 2.24633148545678)}
Let \( \Omega = [-M^2 \pi, M^2 \pi] \). If a signal \( f(t) \) is bandlimited to \( \Omega \), its approximation at scale \( J \) is given by

\[
f(t) \approx P f(t) = \sum_k <f \psi_{0,J,k}, \psi_{0,J,k}>.
\]

If the WTF is an orthonormal basis, and if \( QF = f - Pf \), denote the approximation error (from orthonormality); minimizing \( \|QF\|^2 \) is the same as maximizing \( \|Pf\|^2 \), which is given by [12], [13]

\[
\|Pf\|^2 = \frac{1}{2\pi} \int_{\Omega} |\hat{f}(\omega)|^2 |\hat{\psi}_0(\omega/M^2)|^2 \, d\omega.
\]

Since \( \psi_0(t) \) is determined uniquely by \( h_0 \) and parameterized by \( J(m-1) \) parameters, the optimal design problem becomes an unconstrained optimization problem with that many parameters. An efficient numerical scheme for this optimization (that avoids explicit computation of the integrals in (46)) is given in [12].

We now give an application of the optimal design technique to design smooth WTF's. The essential intuitive idea behind our approach is that the optimal scaling function representing a smooth signal will tend to be smooth. Experimental evidence (tried for different values of \( M, N \) and different choices of \( f(t) \)) corroborates this intuition. Another interesting observation is that for smooth signals, the scaling function is robust to the specific choice of smooth signal (i.e., it does not matter that \( f(t) \) is as long as it is smooth).

Example 4: In this example \( M = 5 \) and \( N = 40 \). The prototype filter corresponding to this design is given in Table VI. The scaling function and its Fourier transform is shown in Fig. 10.

In this example, it can be verified that the partial moments do not match up (even for the first moment), and hence, the scaling function is not differentiable. However, for signal processing applications, the smoothness of the scaling function is adequate.

VI. TOWARDS A NEW DEFINITION OF SMOOTHNESS

Solving nonlinear equations for \( K \) regularity, the method of (weighted) moments minimization, and the optimal design, each has its own application niche. Moreover, each method, under suitable conditions, gives smooth WTF’s. All through this paper, the word smooth has been used loosely, appealing to “intuition” more so than reason. To motivate our definition, a fundamental understanding of why smoothness is needed is crucial. Our discussion, though made in the context of cosine-modulated filter banks and WTF’s, is equally applicable for the general filter bank and WTF design problems.

First, consider the filter bank design problem. It is easy to show that the goal is [18]

\[
\min_\varepsilon \int_{|\omega| > 2M^2 + \varepsilon} |G(\omega)|^2 \, d\omega = \min_\varepsilon E(g),
\]

where \( \varepsilon \) controls the transition width of the filters. If the filter bank is not used in a tree structure, then apparently, there is no reason to consider smoothness, regularity, etc. However, in most applications (like image/speech coding), filter banks are used in a tree structure. The impulse response of the effective filter along, say, the low-pass synthesis branch of such a tree is given by \( H_0(z^{-1}) H_0(z^{-M}) \ldots H_0(z^{-M^J-1}) \), where \( J \) is the depth of the tree. Since the reconstructed image is a superposition of such responses, it is desirable that this response is not too oscillatory. One way to control oscillations is to minimize the total variation of the impulse response.

We therefore define the depth-\( J \) smoothness \( TV_J \) of a filter bank to be the total variation of its depth-\( J \) low-pass filter impulse response. That is

\[
TV_J = \sum_i \|b_J(i) - b_J(i-1)\|.
\]

where \( b_J(i) \) is given by

\[
B_J(z) = H_0(z) H_0(z^M) \ldots H_0(z^{M^J-1}).
\]

Therefore, the filter bank design problem becomes one of minimizing some function of both \( TV_J \) and \( E \). The depth of the tree to be used in the application thus explicitly enters the design process. Many measures of smoothness (like the total variation of first differences of \( B(z) \) etc.,) and combinations thereof can also be considered. An advantage of this method is that (locally) it tries to stretch out kinks in the impulse response and, hence, forces it to be smooth. For all filter banks designed using this method, plots of \( b(\nu) \) (for moderate to large \( J \)) were always smooth. Fig. 11 shows the impulse response of the low-pass filter \( h_0 \) designed by this method for a two-channel, cosine-modulated filter bank with \( N = 16 \). The filter bank was assumed to be used in a five-level tree. The design objective was (notice that other combinations of \( TV_J \) and \( E \) could also be used—in fact, the choice should be dictated by the application)

\[
\min_\varepsilon TV_J \varepsilon.
\]

A direct design based on minimizing \( E \) is also given for comparison. Both examples were designed without the WTF constraint, simply to emphasize that this is a filter bank design approach (i.e., more general than a WTF design). Clearly, the depth-5 smoothness of our new design is better. Moreover, the stop-band attenuation is also better (which also means that the pass-band behavior is better because of the unitariness of the filter bank). This improvement is achieved at the cost of an almost imperceptible increase in the transition width of the filter (see Fig. 11).
Now, consider the design of smooth WTF's. Here again, the total variation of the scaling function is defined to be the smoothness of the WTF. The smoothness of the WTF as defined above is exactly the same as the depth-∞ smoothness $TV_\infty$ of the associated unitary filter bank. Indeed

$$TV_\infty(v_0) = \int_{\text{supp}(v_0)} |\frac{d}{dt}v(t)| \, dt. \tag{50}$$

VII. CONCLUSION

This paper constructs and parameterizes compactly supported multiplicity $M$, cosine-modulated WTF's. In these WTF's, the scaling vectors, which have lengths of the form $2Mm$, uniquely determine the wavelet vectors (unlike general multiplicity $M$ WTF's). The paper obtains an explicit formula for scaling vectors of length $N = 2M$. In many applications, one desires the WTF to be regular. Several numerical techniques for the construction of $K$-regular scaling vectors are introduced: solution of nonlinear equations, the (weighted) moments minimization method, etc. Because $K$-regularity does not imply smoothness (degree of differentiability) of the WTF, a number of smooth $K$-regular designs (particularly for practically important cases of $m$ and $K$) are given. An approximate formula for scaling/wavelet vectors of length $N = 4M$ obtained by moments minimization method (and giving smooth WTF) is also given. Optimal design of cosine-modulated WTF's for signal representation is described and adapted to the design of smooth WTF's. A novel definition of smoothness (that we think is relevant from a signal processing perspective) is proposed. This concept of smoothness is potentially very important in applications with tree structured filter banks. Preliminary designs of filter banks and WTF's based on this approach give good results. This method minimizes undesirable oscillations in the scaling function. All cosine-modulated WTF's constructed are verified to be orthonormal bases. We conjecture that all cosine-modulated WTF's are orthonormal bases. This paper only describes cosine-modulated WTF's with $N = 2Mm$. A natural question becomes: Can this form of restriction on $N$ be released? To answer this question, a comprehensive theory of modulated filter banks (with no assumptions on length of the filters, unitariness of the filter bank, etc.) has been developed, and corresponding to these filters, one obtains modulated filter banks and wavelet frames (tight and otherwise) [29, 30].

REFERENCES

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