A geometric graph is a simple graph $G$, where the vertex-set $V(G)$ is a finite set of points in the plane, and each edge in $E(G)$ is a closed segment whose endpoints belong to $V(G)$. We assume that no three vertices are collinear. A geometric graph is non-crossing if no two edges cross. Two non-crossing geometric graphs are compatible if they have the same vertex set and their union is non-crossing. A matching is non-crossing geometric graph in which every vertex has degree at most one. A matching is perfect if every vertex has degree exactly one. We say that a (perfect) matching is a (perfect) matching of its vertex set. Our focus is on compatible perfect matchings.

We first consider the problem of transforming a given perfect matching into another given perfect matching on the same vertex set. Let $S$ be a set of $n$ points in the plane, with $n$ even. For perfect matchings $M$ and $M'$ of $S$, a transformation between $M$ and $M'$ of length $k$ is a sequence $M = M_0, M_1, \ldots, M_k = M'$ of perfect matchings of $S$, such that $M_i$ is compatible with $M_{i+1}$, for all $i \in \{0, 1, \ldots, k-1\}$. Houle et al. [3] proved that there is a transformation of length $n-2$ between any given pair of perfect matchings of $S$. We improve this bound from linear to logarithmic.

**Theorem 1.** For every set $S$ of $2n$ points in general position, there is a transformation of length $2\lceil \log_2 n \rceil$ between any given pair of perfect matchings of $S$.

The remaining results concern the following conjecture. Two geometric graphs are disjoint if they have no edge in common. A matching is even or odd if the number of edges is even or odd.

**Compatible Matching Conjecture.** For every even perfect matching $M$, there is a perfect matching that is disjoint and compatible with $M$.

Note that the assumption that the given perfect matching is even is necessary. We attack this conjecture using extensions. Let $M$ be a perfect matching with $n$ edges. An extension of $M$ is a set of segments and rays obtained by extending each segment in $M$, in some given order, by rays in both directions. Each ray is extended until it hits another segment, or a previous extension, or the ray goes to infinity. An extension $L$ of $M$ defines a convex subdivision with $n + 1$ cells. Each vertex of $M$ is on the boundary of exactly two cells. The dual multigraph $G$ of $L$ is the (non-geometric) multigraph whose vertices are the cells of this subdivision. For every vertex $v$ of $M$, add an edge to $G$ between the vertices that correspond to the two cells of which $v$ is on the boundary. Thus $G$ has $n + 1$ vertices and $2n$ edges.

An orientation of a (non-geometric) multigraph is even if every vertex has even indegree. Frank et al. [2] proved that a connected multigraph $G$ admits an even orientation if and only if $|E(G)|$ is even.

**Extension Conjecture.** Every even perfect matching $M$ has an extension $L$, such that the associated dual multigraph $G$ admits an even orientation, with the property that whenever a vertex $v$ of $G$ has indegree 2, the two incoming edges at $v$ do not arise from the same segment in $M$.

**Lemma 2.** The Extension Conjecture implies the Compatible Matching Conjecture.

**Proof.** For each oriented edge $xy$ of $G$ corresponding to a vertex $v$ of $M$, assign $v$ to the cell $y$. Since the orientation of $G$ is even, an even number of vertices are assigned to each cell. Moreover, by the final assumption in the Extension Conjecture, if exactly two vertices are assigned to one cell, then the two...
vertices are not adjacent in $M$. It follows that $M$ restricted to the set of vertices that are assigned to each cell has a disjoint compatible perfect matching (contained within the cell). Taking their union, we obtain a perfect matching of the whole set that is disjoint and compatible with $M$.

**Two Trees Conjecture.** Every (even or odd) perfect matching $M$ has an extension $L$, such that the associated dual graph $G$ has an edge-partition into two spanning trees $T_1$ and $T_2$, and for every segment $vw$ of $M$, the edge of $G$ corresponding to $v$ is in a different tree from the edge of $G$ corresponding to $w$.

**Lemma 3.** The Two Trees Conjecture implies the Extension Conjecture.

*Proof.* We may assume that $M$ is even. Thus $G$ has an odd number of vertices, and each $T_i$ has an even number of edges. Hence each $T_i$ has an even orientation. Their union is an even orientation of $G$, such that if a vertex of $G$ has indegree 2, then the two incoming edges are both in $T_1$ or both in $T_2$, and thus arise from distinct segments. Hence the Extension Conjecture is satisfied.

**Theorem 4.** Every even perfect matching $M$ consisting of vertical and horizontal segments has a disjoint compatible perfect matching.

*Proof Sketch.* As illustrated in the figure, first extend each horizontal segment, and then extend each vertical segment. For each horizontal segment, colour the edge of $G$ through the left endpoint red and colour the edge of $G$ through the right endpoint green. For each vertical segment, colour the edge of $G$ through the bottom endpoint red and colour the edge of $G$ through the top endpoint green. We prove that the red and green subgraphs are both spanning trees of $G$. For every segment $vw$, the edge of $G$ passing through $v$ is in a different tree from the edge of $G$ passing through $w$. Thus the Two Trees Conjecture is satisfied. The result follows from Lemmas 2 and 3.

Given that the Compatible Matching Conjecture has remained elusive, it is natural to consider how large a disjoint compatible matching can be guaranteed.

**Theorem 5.** Let $M$ be a perfect matching of a set $S$ of $2n$ points in the plane, with $n$ even. Then there is a matching $M'$ of $S$ with at least $\lfloor (3n - 2)/4 \rfloor$ segments, such that $M$ and $M'$ are compatible and disjoint.

Note that Benbernou et al. [1] made some progress on the Two Trees Conjecture. They proved that every perfect matching has an extension such that the associated dual multigraph $G$ is 2-edge-connected, which is a necessary condition for $G$ to have the desired partition into two trees.

**References**

