Algorithms for area and volume rigidity

Ileana Streinu ∗ Louis Theran †

September 24, 2007

Abstract

Let $P$ be a set of $n$ points in the plane. Laman’s landmark theorem [2] in rigidity theory characterizes exactly which sets of distances determine all $\binom{n}{2}$ distances up to a discrete set of possibilities, generically. Here we ask the same question for sets of triangle areas, answering a question of Whiteley [8].

We define and study area rigidity in dimension 2 and its natural extension to volume rigidity in arbitrary dimensions, proving a Laman-like theorem. The combinatorial characterization is algorithmically tractable by generalizations of pebble games from sparse graphs [3] to sparse hypergraphs [6]. This adds to a very small body of algorithmic rigidity results in dimensions higher than 3.

Keywords: Rigidity theory; computational geometry; sparse hypergraphs; matroids.

1 Introduction and main result

A volume framework is a structure made of fixed volume $(d+1)$-simplices with their vertices on a prescribed point set in $\mathbb{R}^d$; in the planar case $(d = 2)$, we call a volume framework an area framework. In other words, the points of a volume framework are allowed to move in any way that preserves the volumes of all its simplices. A volume framework is volume rigid (or area rigid in the plane) when the only allowed motions preserve the volume of any $(d+1)$-simplex on the point set. We call such a motion a trivial volume-preserving motion. If a volume framework is not volume rigid, it is volume flexible.

Note that volume rigid frameworks are not rigid in the usual (bar-and-joint) sense except when $d = 1$. Figure 1 shows two examples of area frameworks: one is volume rigid and one flexible. Note that volume frameworks do not need to be glued as a simplicial complex.

Figure 1: Area frameworks: (a) area rigid; (b) area flexible (the two area constraints to not fix the area of the triangle indicated by the dashed line). The fixed area triangles are shown in different colors; although the boundaries are shown in solid to improve legibility, the only constraints present are on areas.

We are interested in the following questions, first defined in [3] in the context of bar-and-joint rigidity, about volume frameworks:

Decision Is a volume framework volume rigid?

Extraction Given a volume framework, find a maximum set of independent volume constraints.

Components Given a flexible volume framework, find its maximal volume rigid substructures.

An $s$-graph $G = (V, E)$ is a hypergraph in which every edge contains exactly $s$ vertices; i.e., $V$ is a finite set of $n$ vertices and $E \subset V^s$ is a set of $m$ $s$-edges. A hypergraph $G$ is $(k, \ell)$-sparse if for every edge-induced subgraph $H$ of $G$ on $n'$ vertices, $H$ has at most $kn' - \ell$ edges; sparse $s$-graphs with $kn - \ell$ edges are called tight.

As a combinatorial model for volume frameworks we use an $(s+1)$-graph $G$ embedded on a point set in $\mathbb{R}^d$. In [5], we proved the following combinatorial characterization of generic minimally rigid volume frameworks, which is an analogue of Laman’s theorem [2] for volume rigidity. The case $d = 2$ is exactly Whiteley’s original question.

---

*Computer Science Department, Smith College, Northampton, MA. email: streinu@cs.smith.edu
†Department of Computer Science, University of Massachusetts, Amherst, MA. email: theran@cs.umass.edu
Proposition 1 (Minimally rigid generic volume frameworks [5]). Let $G$ be a $(d+1)$-graph. $G$ is realizable as a generic minimally volume-rigid volume framework if and only if it is $(d,d(d+1)-1)$-tight.

It is worth pointing out that $\ell = k(k+1)-1$ is the largest value of $\ell$ such that the $(k,\ell)$-sparse $(k+1)$-graphs form a matroid. This has algorithmic algorithmic implications that we take advantage of.

Proposition 1 identifies generic volume rigidity as a combinatorial property of the underlying hypergraph, raising the hope of combinatorial algorithms for the original geometric questions. Indeed, our solution is based on the pebble game algorithms from [6]. We describe them in the next section.

2 The pebble game for hypergraphs

Pebble games for hypergraphs are generalizations of the pebble games for sparse graphs [3], which, in turn, generalize Jacobs and Hendrickson’s [1] algorithm for Laman graphs.

Pebble game algorithms are based on a family of hypergraph construction rules indexed by non-negative integer parameters $k$ and $\ell$, which can be intuitively described as a single-player game. The game is played on an oriented hypergraph (each edge is assigned a distinguished tail), which has “pebbles” on some of the vertices. Each of the moves either adds or reorients an edge; the allowed moves are determined by the orientation of the edges and the location of the pebbles. We now describe the $(k,\ell)$-pebble game in terms of its initialization and the allowed moves, shown in Figure 2.

Pebble shift move: Let $v$ be a vertex with at least one pebble on it, and let $e$ be an edge with $v$ as one of its endpoints and with tail $w$. Move the pebble to $w$ and make $v$ the tail of $e$.

We call $H$, taken as an unoriented hypergraph, a pebble game graph. From [6], we have the following characterization.

Proposition 2 (Pebble games and sparse hypergraphs [6]). A hypergraph $G$ is a $(k,\ell)$-pebble game graph if and only if $G$ is $(k,\ell)$-sparse.

Proposition 2, along with the matroidal property of sparse hypergraphs [7, 6] allow us to use the pebble game as the basis of an efficient algorithm for the Extraction problem. Decision is a simple reduction, and Components uses an extension. All of these algorithms are given in detail in [6].

Using a generalization of our union/pair-find data structures [3, 4], our algorithms run in time $O(n^2 + m)$ in the general case. For Decision, a simplification reduces this to $O(n^2 + m)$, which is much better for a sparse input. For volume rigidity, $k = d$, $\ell = d(d+1)-1$, and $s = d+1$.

References


