# Unfolding Convex Polyhedra via Quasigeodesics: Abstract

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## Abstract

We show that cutting shortest paths from every vertex of a convex polyhedron to a simple closed quasigeodesic, and cutting all but a short segment of the quasigeodesic, unfolds the surface to a planar simple polygon.

The full paper is available at `http://arxiv.org/abs/0707.4258v3`.


## 1 Introduction

There are two general methods known to unfold the surface $P$ of any convex polyhedron to a simple polygon in the plane: the source unfolding and the star unfolding. In this note we describe a third method to unfold any convex polyhedron to a simple polygon, answering a question raised in [DO07, p. 307].

A geodesic is a locally shortest path on a smooth surface. A quasigeodesic is a generalization that extends the notion to nondifferentiable, and in particular, to polyhedral surfaces. A *quasigeodesic* $\Gamma$ has $\pi$ total face angle incident to each side at all nonvertex points (just like a geodesic), and has $\leq \pi$ angle to each side where $\Gamma$ passes through a polyhedron vertex. A *closed quasigeodesic* is a closed curve on $P$ that is quasigeodesic throughout its length. Pogorelov showed that any convex polyhedron $P$ has at least three simple closed quasigeodesics [Pog49], extending the celebrated earlier result of Lyusternik-Schnirelmann showing that the same holds for geodesics on differentiable convex surfaces.

## 2 Quasigeodesic Unfolding

We now describe the unfolding procedure, which consists of three main steps after identifying a simple closed quasigeodesic $\mathcal{Q}$, which partitions $P$ into two "halves" $P_1$ and $P_2$:

1. Select shortest paths $\mathrm{sp}(v)$ from each $v \in P_i$ to $\mathcal{Q}$.
2. Cut along $\mathrm{sp}(v)$ and flatten each half.
3. Cut along $\mathcal{Q}$, joining the two halves at an uncut segment $s$.

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We will use a cube as an illustrative example throughout. Let $\mathcal{Q}$ be the closed quasigeodesic $(v_0, v_5, v_7)$ on the surface of the cube labeled as in Figure 1. We will call

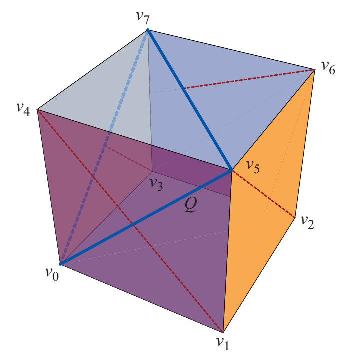


Figure 1: Cube and quasigeodesic $\mathcal{Q} = (v_0, v_5, v_7)$. Shortest paths $\mathrm{sp}(v_i)$ marked in red.

the right half (including $v_1$) $P_1$, and the left half $P_2$.

### 2.1 Shortest Paths.

The unfolding begins by cutting shortest paths from each vertex to $\mathcal{Q}$. For any vertex $v \in P_i$ (and not on $\mathcal{Q}$), select one shortest path $\mathrm{sp}(v)$ from $v$ to a point $q(v) \in \mathcal{Q}$ that realizes the shortest distance in $P_i$ between $v$ and (any point of) $\mathcal{Q}$. (There could be several shortest paths tied for this minimum; $\mathrm{sp}(v)$ is chosen arbitrarily among these.)

A central fact that enables the construction is this key lemma from [IIV07, Cor. 1]:

**Lemma 1** *Let $\mathcal{Q}$ be a simple closed quasigeodesic on a convex surface $S$, and $p$ any point of $S$ not on $\mathcal{Q}$. Then* $\mathrm{sp}(p)$ *is the unique shortest path from $p$ to $q(p)$, and it is orthogonal to $\mathcal{Q}$.*

A second fact we need concerning these shortest paths is that they are disjoint:

**Lemma 2** *Any two shortest paths* $\mathrm{sp}(v_1)$ *and* $\mathrm{sp}(v_2)$ *are disjoint, for distinct vertices $v_1, v_2 \in P_1$.*

### 2.2 Flattening the Halves.

The next step is to flatten each half $P_1$ and $P_2$ (independently) by suturing in additional surface along each $\mathrm{sp}(v)$ path. Let $L$ be the length $|\mathrm{sp}(v)|$ of a shortest path, and let $\gamma = \gamma(v) > 0$ be the curvature at $v$. We glue into $\mathrm{sp}(v) = (v, q(v))$ the isosceles triangle $\triangle$ with apex angle $\gamma$ gluing to $v$, and incident sides of length $L$ gluing along $\mathrm{sp}(v)$. This is illustrated in Figure 2, where we show the faces incident to $\mathcal{Q}$ in a planar development in (a) and (c), and after gluing in the triangles in (b) and (d). We display this in the plane for convenience of presentation; the triangle insertion should be viewed as operations on the manifolds $P_1$ and $P_2$, each independently.

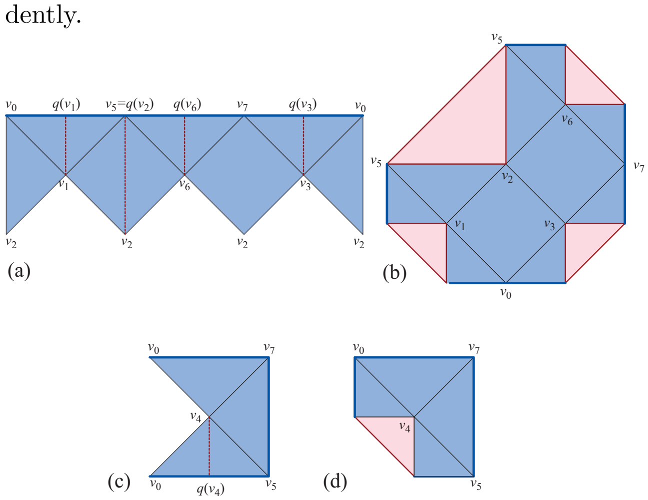


Figure 2: (a,b) Flattening $P_1$ by insertion of triangles (light) along the shortest paths $\mathrm{sp}(v_i) = (v_i, q(v_i))$. (c,d) Flattening $P_2$.

This procedure only works if $\gamma < \pi$, for $\gamma$ becomes the apex of the inserted triangle $\triangle$. If $\gamma \geq \pi$, we glue in two triangles of apex angle $\gamma/2$, both with their apexes at $v$. Slightly abusing notation, we use $\triangle$ to represent these two triangles together.

Now, because $\gamma$ is the curvature (angle deficit) at $v$, gluing in $\triangle$ there flattens $v$ to have total incident angle $2\pi$. Thus $v$ disappears from $P_i$ (and two new vertices are created along $\mathcal{Q}$).

Call the new manifolds with boundary after insertion of all $\triangle$'s $P_1'$ and $P_2'$.

**Lemma 3** *$P_i'$ is a planar convex polygon.*

See Figure 2(b,d). Note that, when the total curvature in $P_i$ is $2\pi$ then the straight development of $\mathcal{Q}$ is turned $2\pi$ by the $\triangle$ insertions, as in (b) of the figure. When the total curvature in $P_i$ is $< 2\pi$, the development of $\mathcal{Q}$ is not straight, but the $\triangle$ insertions turn it exactly the additional amount needed to close it to $2\pi$, as in (d) of the figure.

### 2.3 Joining the Halves.

The third and final step of the unfolding procedure is to select a segment $s$ of $\mathcal{Q}$ whose interior contains no $v$ nor $q(v)$, and to cut all of $\mathcal{Q}$ except for $s$. In our example, we choose $s = (v_5, q(v_6))$. Then lay out $s$ horizontal in the plane, and arrange the convex polygons $P_i'$ above and below, joined along $s$. Because they are convex and separated by the line through $s$, they do not overlap. Removing the inserted triangles $\triangle$ results in an unfolding of the original $P$. See Figure 3.

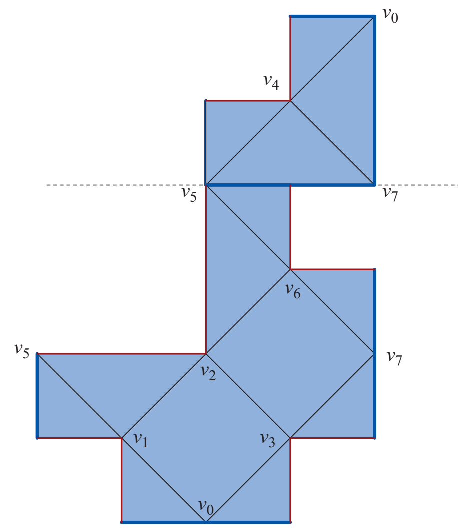


Figure 3: Unfolding of the cube shown in Figure 1. The top and bottom "halves" derive from the convex polygons shown in Fig. 2(d) [top half, rotated 90°] and (b) [bottom half].

It should be clear now that this procedure works for any convex polyhedron:

**Theorem 4** *Let $\mathcal{Q}$ be a simple closed quasigeodesic on a convex polyhedron surface $P$. Cutting shortest paths from every vertex to $\mathcal{Q}$, and cutting all but a segment $s$ of $\mathcal{Q}$ free of vertices and shortest path endpoints, unfolds $P$ to a simple planar polygon.*

As $s$ is choosen arbitrarily, the position of $P_i'$ with respect to the support line of $s$ depends on this choice.

## References

[DO07] Erik D. Demaine and Joseph O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra.* Cambridge University Press, July 2007. `http://www.gfalop.org`.

[IIV07] Kouki Ieiri, Jin-ichi Itoh, and Costin Vîlcu. Quasigeodesics and farthest points on convex surfaces. Submitted, 2007.

[Pog49] Aleksei V. Pogorelov. Quasi-geodesic lines on a convex surface. *Mat. Sb.*, 25(62):275–306, 1949. English transl., *Amer. Math. Soc. Transl.* 74, 1952.