Minimum Outer Layer and Zone Complexity

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1 Introduction

In an arrangement of lines in the Euclidean plane (hereafter a Euclidean arrangement) by outer layer we mean the collection of all unbounded cells. We refer to the number of segments bounding the unbounded cells as the outer layer complexity of the arrangement. For example, the outer layer complexity of the arrangement in Figure 1 is 16.

Figure 1. An arrangement of 6 lines with outer layer complexity 16. There are two pairs of parallel lines. The bounding segments of the outer layer are highlighted.

For a given number \( n \) we consider all Euclidean arrangements of \( n \) lines in a certain class and ask what the minimum complexity of the outer layer may be. In this pursuit, we rule out the two “trivial” arrangements (1) where all lines pass through a common point and (2) where all lines are parallel.

2 Background

In the case of maximum complexity of the outer layer, if we introduce the usual general position assumption (no two lines parallel, no three lines passing through a common point), the problem reduces to that of finding the maximum complexity of the zone of a line \( \ell \). In the classical definition of the complexity of the zone of a line \( \ell \) is the total number of edges of faces in the arrangement (minus the line \( \ell \)), whose faces intersect \( \ell \). See Figure 2 to appreciate the distinction.

Figure 2. An arrangement of 3 lines with the addition of a (dashed) line \( \ell \). In the classical definition of the complexity of the zone of \( \ell \), the two sides of each line crossing \( \ell \) are counted separately since they bound adjacent faces of the arrangement. On the other hand, in our definition of complexity, for each line crossing \( \ell \) we count the segment above \( \ell \) and the segment below \( \ell \) as distinct. Hence the two definitions are equivalent and under either definition the complexity of the above arrangement is 10.

When considering the complexity of the zone, our definition of complexity differs from the classical one by how we count the segments which cross the line \( \ell \). Although the perspective is slightly different, in either case, each line (other than \( \ell \)) contributes precisely two segments which intersect \( \ell \). Thus the answer to the question of the maximum complexity of the outer layer, for lines in general position, is answered, up to a constant, by the theorems giving best possible constants in the Zone Theorem. In particular, Bern, et al. [1] showed the complexity to be at most \( 5.5n \) and gave examples to show that the bound is tight up to an additive constant.

If we don’t make a general position assumption, however, our definition of outer layer complexity differs from the intuitive notion of zone complexity. We are faced with the circumstance that vertices of the arrangement lie on the line \( \ell \) and we must count the edges of all faces associated with such vertices in the complexity. However, the edges emanating from the vertex are counted just once in the outer layer complexity, but twice, once for each face they bound, in the zone complexity.
3 Results

In the current work, we use the following result on maximum numbers of wedges [2] to get a lower bound on the complexity of the outer layer for non-trivial Euclidean arrangements without general position assumptions, i.e., we allow parallel lines and multiple lines passing through a point. A wedge is an unbounded two-edged face in a Euclidean arrangement of lines.

**Lemma 1** In a non-trivial Euclidean arrangement of $n$ lines (not all lines parallel and not all lines passing through a common point), there are at most $\left\lceil \frac{4n}{3} \right\rceil$ wedges and this bound is tight for all $n$.

Our first result is then:

**Theorem 2** For $n \geq 3$ the minimum possible outer layer complexity of non-trivial Euclidean arrangements of $n$ lines is $\frac{3n}{2} + O(1)$, and in particular, at most $\min(\left\lfloor \frac{4n}{3} \right\rfloor + 2, 3n - 2)$. For $n \equiv 3 \, (\text{mod} \, 6)$, $n \geq 9$, the minimum possible outer layer complexity is exactly $\frac{8n}{3}$.

Tight examples for $n = 9, 15$ are given in Figure 3. The idea of the proof is very simple. Each line contributes two unbounded edges to the outer layer complexity. As we work our way counterclockwise, say, around the unbounded edges, we encounter $2n$ “slots” between these edges. By the Lemma, at most $\left\lfloor \frac{4n}{3} \right\rfloor$ of these can be wedges. The best one can do at the remaining $\left\lceil \frac{4n}{3} \right\rceil$ slots is have an unbounded triangle. For the cases $n \equiv 3 \, (\text{mod} \, 6)$, $n \geq 9$ one can always find such an arrangement. We then add 5 lines one at a time in such a way that the total complexity never exceeds $\left\lfloor \frac{8n}{3} \right\rfloor + 2$. The $3n - 2$ term comes in because in our analysis we have tacitly assumed that a bounded edge contributes to at most one unbounded face (triangle). The peculiar case in which an unbounded edge contributes to two unbounded faces happens only if the arrangement contains $n - 1$ parallel lines and an additional line not parallel to the others. In this case we obtain an arrangement with total complexity $2(n - 1) + n = 3n - 2$.

Hence the extra term. Of course, $3n - 2 < \left\lceil \frac{8n}{3} \right\rceil + 2$ only for relatively small $n$.

If we consider arrangements in general position we obtain a cleaner conclusion. In this case the lower bound on the complexity of the outer layer translates directly to a lower bound on the complexity of the zone of a line.

**Theorem 3** Any Euclidean arrangement of $n \geq 3$ lines in general position has outer layer complexity at least $4n - 2$ and this bound is achievable for every such $n$.

Questions of minimum complexity can also be asked in higher dimensions, where, for example in three dimensions, complexity is measured in terms of the number of bounding faces. The upper bound for the general position case is again covered by the constants from the higher dimensional Zone Theorem, but for the lower bound, we do not even have an analog of the maximum wedges result, so that case and its general position analog are completely open.

The lower bound problem for planes in general position seems a priori to be a lot harder than the two dimensional analog. Lines in non-trivial Euclidean arrangements each have two unbounded edges enabling us to analyze such problems in terms of the $2n$ “slots” between unbounded edges. However, planes in general position have $n$ unbounded faces at their “tops,” $n$ unbounded faces at their “bottoms,” and $2n - 4$ additional unbounded faces on their “sides.” Further, an unbounded cell can have up to $n$ unbounded faces so any notion of slot is much more complicated.

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**References**
