Ordinary Points in Polygons

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1 Introduction

The Sylvester-Gallai Theorem [1, 4, 7] tells us that a finite collection of lines in the projective plane, not all passing through a single point, must have ordinary points - points of intersection of precisely two lines. Much work has gone into trying to understand, for a given \( n \), what the minimum number of ordinary points can be amongst all non-trivial arrangements of \( n \) lines [2,5] - indeed this problem has turned out to be quite difficult. Csima and Sawyer [2] have shown that for \( n \neq 7 \) there must be at least \( \frac{n^2}{11} \) ordinary points and it has been widely conjectured (beginning with [3]) that there for \( n \neq 7, 13 \) that there must be at least \( \frac{n}{5} \) ordinary points.

Melchior [6] showed that the Sylvester-Gallai Theorem is a consequence of Euler’s formula regarding the facial structure of line or pseudoline arrangements in the plane. Euler’s formula also holds within any polygon determined by the lines of an arrangement. However, at least in its most obvious form, there is no Sylvester-Gallai Theorem for the lines intersecting in generic polygons. In this note we formulate and sketch the proofs of variants of the Sylvester-Gallai Theorem suitable for lines intersecting in simple polygons with up to five edges. By examining extremal examples of ordinary points in polygons and then studying how these polygons can be pieced together to get line or pseudoline arrangements we hope to develop new methods for attacking the ordinary points in arrangements problem.

2 Background

Given an arrangement \( \mathcal{A} \) of lines in the projective plane with \( V \) vertices, \( E \) edges, and \( F \) faces, Euler’s formula says that

\[
V - E + F = 1.
\]

Following Melchior, we put

\[
t_j = \text{number of vertices where } j \text{ lines cross},
\]

\[
p_k = \text{number of faces surrounded by } k \text{ edges},
\]

and write

\[
Y = \sum_{j \geq 2} (3-j)t_j + \sum_{k \geq 3} (3-k)p_k. \tag{2}
\]

Combining

\[
\sum_{j \geq 2} t_j = V, \quad \sum_{k \geq 3} p_k = F, \tag{3}
\]

with the easily observed relations

\[
\sum_{j \geq 2} j t_j = E, \quad \sum_{k \geq 3} kp_k = 2E, \tag{4}
\]

gives

\[
Y = (3V - E) + (3F - 2E) = 3(V - E + F) = 3. \tag{5}
\]

Now, looking at (2) we note that only the coefficient of the \( t_2 \) term, which is 1, is positive, so we must have \( t_2 \geq 3 \), establishing Sylvester’s Theorem (in fact a slightly stronger conclusion, namely that there have to be at least three ordinary points).

3 Results

Theorem 1 Consider any triangle in an arrangement and the vertices formed inside the triangle by lines which intersect the interior of the triangle. Amongst these vertices and lines only, there must be at least 2 ordinary points.

To better understand the statement of the Theorem, see Figure 1.

Proof (Sketch). Let \( T \) be an arbitrary triangle in an arrangement. In any polygon, the projective form of Euler’s relation (1) holds and we would like to use this to tell us something about the familiar expression

\[
Y_T = \sum_{j \geq 2} (3-j)t_j + \sum_{k \geq 3} (3-k)p_k \tag{6}
\]

where \( t_j \) now denotes the number of \( j \)-crossings in \( T \) and \( p_k \) denotes the number of \( k \)-gons in \( T \).

We still have

\[
\sum_{j \geq 2} t_j = V, \quad \sum_{k \geq 3} p_k = F. \tag{7}
\]
Figure 2. The finite triangle $T = \triangle(p, q, r)$ with external faces $F_1, ..., F_7$ and external edges $E_1, ..., E_8$ as described in the proof of Theorem 1.

However edges along the border of $T$ are not shared by two faces, and edges don’t extend outward from the borders of $T$ so we don’t have either of the relations in (4).

However, let us declare that we have a single “external face” wherever there is an edge lying along the border of $T$ and similarly say that every line which is either a defining line of $T$ or intersects the interior of $T$ has a single “external edge” on the outside of $T$. See Figure 2. If we let

$$\begin{align*}
F_{\text{ext}} &= \text{number of External Faces} \\
E_{\text{ext}} &= \text{number of External Edges}
\end{align*}$$

then we have

$$\sum_{k \geq 3} kp_k + F_{\text{ext}} = 2E, \quad \text{(10)}$$
$$\sum_{j \geq 2} jt_j = E + E_{\text{ext}}. \quad \text{(11)}$$

Let us also define

$$\text{Excess Crossings} = \sum_{j \geq 2} (j - 3)t_j.$$  

It is not terribly difficult to establish that

$$E_{\text{ext}} \leq F_{\text{ext}} + \text{Excess Crossings} + 1. \quad \text{(12)}$$

Substituting (1), (7), (10), (11) and (12) into (6) gives

$$Y_T = 3V - (E + F_{\text{ext}}) + 3F - (2E - F_{\text{ext}}) = 3 + F_{\text{ext}} - E_{\text{ext}} \geq 2 - \text{Excess Crossings}$$

i.e.

$$\sum_{j \geq 2} (3 - j)t_j + \sum_{k \geq 3} (3 - k)p_k \geq 2 - \text{Excess Crossings} \quad \text{(13)}$$

so that $t_2 \geq 2$.

The same sort of argument can be used to prove the following:

**Theorem 2** If a triangle $T$ as considered in Theorem 1 has only one $(3^+)$-crossing amongst its defining vertices and we consider just the lines of $T$ together with lines intersecting the interior of $T$ then $T$ must contain an ordinary point in addition to its two ordinary vertices.

**Theorem 3** Consider any simple 4- or 5-gon $P$ in an arrangement and the vertices inside $P$ formed by $P$ and lines intersecting the interior of $P$. Amongst these vertices and lines only, there must be at least 1 ordinary point.

See the Figure 3 for tight examples. The conclusion of Theorem 2 does not hold for 4-gons and higher. Similarly, there are 6-gons (and clearly $2n$-gons for all $n > 3$) without ordinary points. See Figure 4.

The methods we have developed do not seem powerful enough to decide whether or not there are ordinary points in odd $n$-gons for $n \geq 7$. Questions regarding lower bounds on the number of ordinary points in polygons as a function of the number of lines may also be pursued.

**References**


