ON THE NUMBER OF EUCLIDEAN
ORDINARY POINTS FOR LINES IN THE
PLANE

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Abstract

Given an arrangement of $n$ not all coincident, not all parallel lines in the (projective or) Euclidean plane we have earlier shown that there must be at least $(5n + 6)/39$ Euclidean ordinary points. We improve this result to show there must be at least $n/6$ Euclidean ordinary points.

1 Sylvester’s problem in the Euclidean plane

A classical theorem of Sylvester and Gallai states that given a set of $n$ not all collinear points in the plane, there must be at least one line which passes through exactly two of the points. The theorem has a corresponding dual statement, namely that any collection of $n$ lines in the projective plane has at least one point where precisely two of the lines intersect. We call such a point an ordinary point. The Theorem of Sylvester and Gallai is known to follow from Euler’s formula for projective arrangements. See

Many proofs of the Sylvester-Gallai theorem are known, the first of which was given by Gallai in 1944 [4]. Following the proof of the Sylvester-Gallai theorem, attention turned to giving a lower bound on the number of such ordinary points. In 1958 Kelly and Moser [5] proved that there must be $3n/7$ ordinary points, and then in 1993 Csima and Sawyer [2] proved that as long as $n \neq 7$, there must be at least $6n/13$ ordinary points.

Recently, Lenchner [6, 7] considered the following variant of Sylvester’s problem: In an arrangement of $n$ lines in the Euclidean plane, not all of which all of which are parallel and not of which pass through a common point, must there be a (Euclidean) ordinary point? In [7] it was pointed out that a positive answer to this question does not follow from Euler’s formula, but that a bound of $(5n + 6)/39$ does follow as a consequence of the bound of Csima and Sawyer. In this paper, we improve the $(5n + 6)/39$ bound to $n/6$ without using the Csima and Sawyer theorem.

2 The $(5n + 6)/39$ Result

**Theorem 1** In an arrangement of $n$ not all collinear, not all coincident lines in the Euclidean plane, there must be at least $(5n + 6)/39$ Euclidean ordinary points.

**Proof.** We consider the problem embedded in the real projective plane, where the theorem of Csima and Sawyer [2] says that there must be at least $6n/13$ ordinary points except when $n = 7$. The $n = 7$ case of the Theorem is handled by Lemma 4 from [7] as a simple consequence of the theory of wedges.

If our result were false then more than $\lceil \frac{6n}{13} - \frac{5n + 6}{39} \rceil = \lceil \frac{n}{3} - \frac{2}{13} \rceil$ of these ordinary points would have to lie on the line at infinity. In other words there would have to be at least $\lceil \frac{n}{3} - \frac{2}{13} \rceil$ pairs of parallel lines. To this arrangement add the line at infinity. This
“kills off” the at least \( \lceil \frac{n}{3} - \frac{2}{13} \rceil \) ordinary points and creates at most \( \lfloor n - \frac{2n}{3} + \frac{4}{13} \rfloor = \lfloor \frac{n}{3} + \frac{4}{13} \rfloor \) new ordinary points.

By the theorem of Csima and Sawyer applied to the new arrangement (as long as \( n \neq 6 \), a case we cover at the end) we have at least \( \lceil \frac{6(n+1)}{13} \rceil \) ordinary points. But then there must have been at least \( \lceil \frac{6(n+1)}{13} - \frac{n}{3} - \frac{4}{13} \rceil = \lceil \frac{5n+6}{39} \rceil \) finite ordinary points earlier, contradicting our initial assumption. The result is thus proved.

If \( n = 6 \) the theorem claims that there is at least one finite ordinary point. By the theorem of Kelly and Moser, we know that there are at least 3 total ordinary points. If all such points were on the line at infinity the implication would be that we have 3 pairs of parallel lines. Adding a seventh line at infinity would yield a projective arrangement without ordinary points, a clear impossibility. The theorem follows. □

The algebra that allows one to arrive at the lower bound \( \frac{(5n+6)}{39} \) is described in [6].

## 3 Improving the bound to \( n/6 \)

**Definition 2** Say that an ordinary point \( p \) is **attached** to a line \( \ell \), not containing \( p \), if \( \ell \) together with two lines crossing at \( p \) form a (possibly infinite) triangular cell of the arrangement.

Figure 1 illustrates a line \( \ell \) and its attached points. In this article we focus on finite attached points. Kelly and Moser [5] used the notion of attached points together with a double counting argument to obtain their \( 3n/7 \) bound. Csima and Sawyer [2] added an additional, though highly non-obvious, observation about attached points to those of Kelly and Moser to obtain their \( 6n/13 \) bound. The following simple lemma is used in both papers:

**Lemma 3** In any arrangement of lines, an ordinary point can have at most 4 lines counting that point as an attachment.
Figure 1: An example of a line $\ell$ with four ordinary points attached. The lines $\ell$ and $k$ are drawn to be parallel. The rightmost attached point is attached via an infinite triangle.

**Proof.** An ordinary point is contained in 2 crossing lines, and hence a vertex of 4 faces; it can therefore be attached to at most 4 lines. □

Our central lemma is the following:

**Lemma 4** Let $A$ be a Euclidean arrangement of $n$ lines, with not all lines parallel and not all lines passing through a common point. Then if a line $\ell \in A$ does not contain an ordinary point, then it must have at least one (Euclidean) ordinary point attached to it.

**Proof.** If all the Euclidean vertices are on a single line, then all but that line must be parallel, and all vertices are ordinary. There is thus no line without Euclidean ordinary points.

Thus let $\ell \in A$ be a line without Euclidean ordinary points and let $x$ be the closest vertex to $\ell$, and rightmost if there are several such vertices. We argue that $x$ must be ordinary. In that case, the triangle defined by $\ell$ and the two lines through $x$ must be a cell of the arrangement (possibly infinite if one of those lines is parallel to $\ell$) and so $x$ is attached to $\ell$. If $x$ is not ordinary, then there are at least three lines through $x$, let us call them $\ell_1, \ell_2$ and $\ell_3$, with $\ell_3$ possibly parallel to $\ell$, and $\ell_2$ intersecting $\ell$ between $\ell_1$ and $\ell_3$ (or to the right of $\ell_1$ if $\ell_3$ is parallel). Then
the intersection $y$ of $\ell_2$ and $\ell$ must be non-ordinary, yet any line through it must intersect $\ell_1$ or $\ell_3$ in a point that is closer to $\ell$ than $x$, or to the right of $x$ on $\ell_3$ if $\ell_3$ is parallel to $\ell$, in either case a contradiction. $\Box$

**Theorem 5** Let $\mathcal{A}$ be a Euclidean arrangement of $n$ lines, with not all lines parallel and not all lines passing through a common point. Then $\mathcal{A}$ has at least $n/6$ (Euclidean) ordinary points.

**Proof.** Let $k_i$ denote the number of lines of $\mathcal{A}$ containing exactly $i$ Euclidean ordinary points, and suppose that there are fewer than $n/6$ Euclidean ordinary points in total. Then we have

$$ \sum_{i \geq 1} ik_i < \frac{n}{3} $$

since the sum on the left counts each ordinary point twice.

Also,

$$ \sum_{i \geq 0} k_i = n $$

so that

$$ \sum_{i \geq 0} k_i > \sum_{i \geq 1} 3ik_i $$

so

$$ k_0 > \sum_{i \geq 1} (3i - 1)k_i. $$

But also there are at most 4 lines counting a given ordinary point as an attachment (possibly via an infinite triangle), so that if $1 + \epsilon_0$ denotes the average number of Euclidean attached points for lines with no Euclidean ordinary points (Lemma 4), and $\epsilon_i$ denotes the average number of Euclidean attached points for lines with $i \geq 1$ Euclidean points, we have, for $\epsilon_i \geq 0$, $\forall i \geq 0$,

$$ (1 + \epsilon_0)k_0 + \sum_{i \geq 1} \epsilon_i k_i \leq 2 \sum_{i \geq 1} ik_i $$

(5)
i.e.

\[ k_0 \leq \sum_{i \geq 1} 2i k_i. \] \hspace{1cm} (6)

But \( 3i - 1 \geq 2i \) for \( i \geq 1 \), so equations (4) and (6) cannot simultaneously hold and the theorem follows. \( \Box \)

In projective arrangements, since the choice of which is the line at infinity is completely arbitrary, we have the following immediate corollary:

**Corollary 6** Let \( \mathcal{A} \) be a projective arrangement of \( n \) lines, not all of which pass through a common point. Then there are at least \( n/6 \) ordinary points off any line which is not part of the arrangement.

Slightly less obvious is the following:

**Corollary 7** Let \( \mathcal{A} \) be a projective arrangement of \( n \) lines, no \( n - 1 \) of which pass through a common point. Then there are at least \( (n - 1)/6 \) ordinary points off any line in the arrangement.

*Proof.* Let \( \ell \in \mathcal{A} \) and consider the arrangement with \( \ell \) removed. By the assumption about no \( n - 1 \) of the lines passing through a common point, we may apply the previous Corollary to conclude that there are at least \( (n - 1)/6 \) ordinary points off of \( \ell \), points which are ordinary with or without \( \ell \). \( \Box \)

### 4 Concluding Remarks

The big open conjecture in the classical Sylvester case, where we consider projective lines and projective ordinary points, is that except for \( n = 7, 13 \) that there must be at least \( n/2 \) ordinary points. If this conjecture were true, then the methods of section 2 would immediately imply the \( n/6 \) bound obtained in section 3. Thus the \( n/6 \) result is in some sense stronger than
the Csima and Sawyer bound. A natural question is whether the $n/6$ bound can be used to strengthen Csima and Sawyer’s $6n/13$ bound. We think not, since the argument of section 2 involves a double application of the Csima and Sawyer bound, and probably provides a fair amount of overkill. There are arrangements with exactly $n/2$ ordinary points known for every even $n$, however if $n$ is odd and $n \neq 7, 13$ then the worst known cases have, respectively, $3(n - 1)/4$ ordinary points if $n \equiv 1 \pmod{4}$ and $3(n - 3)/4$ ordinary points if $n \equiv 3 \pmod{4}$. These families of arrangements are due to Böröczky; see [1]. In the double application of the Csima and Sawyer bound we are applying the bound for consecutive values of $n$, hence once for odd $n$ and once for even $n$.

We thus anticipate that the $n/6$ bound can be improved using new insights and perhaps such an improved bound could in turn lead to a proof of the $n/2$ conjecture.

References


