Line Search Filter Methods for Nonlinear Programming: Local Convergence

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Abstract

A line search method is proposed for nonlinear programming using Fletcher and Leyffer's filter method, which replaces the traditional merit function. A simple modification of the method proposed in a companion paper [14] introducing second order correction steps is presented. It is shown that the proposed method does not suffer from the Maratos effect, so that fast local convergence to second order sufficient local solutions is achieved.

Keywords: nonlinear programming – nonconvex constrained optimization – filter method – line search – local convergence – Maratos effect – second order correction

1 Introduction

Recently, Fletcher and Leyffer [7] proposed filter trust region methods, offering an alternative to merit functions, as a tool to guarantee global convergence in algorithms for nonlinear programming (NLP). The underlying concept is that trial points are accepted if they improve the objective function *or* improve the constraint violation instead of a merit function. In a companion paper [14] we propose and analyze a filter line search method which can be applied to equality constrained nonlinear programs, as well as problems with nonlinear equality and bound constraints using active set SQP methods and barrier interior point methods.

In this paper we discuss the local convergence properties of the filter line search algorithm proposed in [14]. As mentioned by Fletcher and Leyffer [7], the filter approach can suffer from the so-called Maratos effect [10]. The Maratos effect occurs if, arbitrarily close to a strict local solution of the NLP (1), a full Newton step increases *both* the objective function and the constraint violation, and is therefore rejected by the line search, even though it could be a very good step toward the solution. This can result in poor local convergence behavior. As a remedy, Fletcher and Leyffer propose to improve the search direction, if the full step is rejected, by means of a second order correction which aims to further reduce infeasibility. In this paper we show that this modification is indeed able to prevent the Maratos effect.

Ulbrich [13] has recently presented a trust region filter method using the Lagrangian function (instead of the objective function) as one of the measures in the filter (similar to what we propose in our companion paper [14]). In [13], Ulbrich shows fast local convergence without second order correction steps.

The paper is organized as follows. In order to keep the analysis simple, we focus first only on the easiest case of equality constrained optimization problems. In Section 2 we revisit the filter line

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search procedure from the companion paper [14]. Section 3 states the modified filter line search algorithm including second order correction steps. The local convergence analysis is presented in Section 4. In Section 5 we briefly discuss how this approach can be applied to a line search and a trust region filter SQP method to handle inequality constrained problems.

Notation. We denote the i-th component of a vector $v \in \mathbb{R}^n$ by $v^{(i)}$. Norms $\|\cdot\|$ denote a fixed vector norm and its compatible matrix norm. We denote by $O(t_k)$ a sequence $\{v_k\}$ satisfying $\|v_k\| \leq \beta\ t_k$ for some constant $\beta > 0$ independent of k, and by $o(t_k)$ a sequence $\{v_k\}$ satisfying $\|v_k\| \leq \beta_k t_k$ for some positive sequence $\{\beta_k\}$ with $\lim_k \beta_k = 0$.

2 A Line Search Filter Method

The proposed algorithm is a filter line search algorithm for solving nonlinear optimization problems of the form

$$\min_{x \in \mathbb{R}^n} \quad f(x) \tag{1a}$$

$$\text{subject to} \quad c(x) = 0 \tag{1b}$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ and the equality constraints $c : \mathbb{R}^n \to \mathbb{R}^m$ with $m < n$ are twice continuously differentiable. The Karush-Kuhn-Tucker (KKT) conditions for this problem are given by

$$g(x) + A(x)\lambda = 0 \tag{2a}$$

$$c(x) = 0 \tag{2b}$$

with the Lagrangian multipliers λ, where $g(x) := \nabla f(x)$ and $A(x) := \nabla c(x)$. Under suitable *constraint qualifications*, such as linear independence of the constraint gradients $\nabla c(x)$, these are the first order optimality conditions for (1) (see e.g. [12]).

Given a starting point x_0, the proposed line search algorithm generates a sequence of improved estimates x_k of the solution for the NLP (1). For this purpose in each iteration k a search direction d_k is computed from the linearization of the KKT conditions (2) at x_k,

$$\begin{bmatrix} H_k & A_k \\ A_k^T & 0 \end{bmatrix} \begin{pmatrix} d_k \\ \lambda_k^+ \end{pmatrix} = - \begin{pmatrix} g_k \\ c_k \end{pmatrix}. \tag{3}$$

Here, $A_k := A(x_k)$, $g_k := g(x_k)$, $c_k := c(x_k)$. The symmetric matrix H_k denotes the Hessian $\nabla^2_{xx}\mathcal{L}(x_k, \lambda_k)$ of the Lagrangian

$$\mathcal{L}(x, \lambda) := f(x) + c(x)^T \lambda \tag{4}$$

of the NLP (1), or an approximation to this Hessian. The vector λ_k is some estimate of the optimal multipliers corresponding to the equality constraints (1b), and λ_k^+ in (3) can be used to determine a new estimate λ_{k+1} for the next iteration. In the context of this paper the particular choice of λ_k is not important. As is common for many line search methods, we assume that the projection of the Hessian approximation H_k onto the null space of the constraint Jacobian is uniformly positive definite to ensure certain descent properties.

After a search direction d_k has been computed, a step size $\alpha_k \in (0, 1]$ is determined in order to obtain the next iterate

$$x_{k+1} := x_k + \alpha_k d_k. \tag{5}$$

In the companion paper [14] we propose backtracking line search procedure, where a decreasing sequence of step sizes $\alpha_{k,l} \in (0, 1]$ $(l = 0, 1, 2, \dots)$ with $\lim_l \alpha_{k,l} = 0$ is tried until an acceptance criterion is satisfied. The procedure that decides which trial step size is accepted is a "filter method." In the remainder of this section we only briefly revisit this approach; the detailed motivation can be found in [14]. The formal statement of the algorithm is presented in Section 3.

Filter methods were originally proposed by Fletcher and Leyffer [7]. The basic idea behind this approach is to interpret the optimization problem (1) as a bi-objective optimization problem with the two goals of minimizing the objective function $f(x)$ and the constraint violation $\theta(x) := \|c(x)\|$ (with a certain emphasis on the latter quantity). Following this paradigm, we might consider a trial point $x_k(\alpha_{k,l}) := x_k + \alpha_{k,l} d_k$ during the line search to be acceptable, if it leads to sufficient progress toward either goal compared to the current iterate, i.e. if

$$\theta(x_k(\alpha_{k,l})) \leq (1-\gamma_\theta)\theta(x_k) \tag{6a}$$

$$\text{or} \quad f(x_k(\alpha_{k,l})) \leq f(x_k) - \gamma_f \theta(x_k) \tag{6b}$$

holds for fixed constants $\gamma_\theta, \gamma_f \in (0,1)$. However, the above criterion is replaced by requiring sufficient progress in the objective function, whenever the following "switching condition"

$$g_k^T d_k < 0 \qquad \text{and} \qquad \alpha_{k,l}[-g_k^T d_k]^{s_f} > \delta\,[\theta(x_k)]^{s_\theta} \tag{7}$$

with constants $\delta > 0, s_\theta > 1, s_f > 2s_\theta$ holds[1]. If (7) is true for the current step size $\alpha_{k,l}$, the trial point has to satisfy the Armijo condition

$$f(x_k(\alpha_{k,l})) \leq f(x_k) + \eta_f \alpha_{k,l} g_k^T d_k, \tag{8}$$

instead of (6), in order to be acceptable. Here, $\eta_f \in (0, \frac{1}{2})$ is a constant. Since the projection of the matrix H_k in (3) onto the null space of A_k^T is uniformly positive definite, it can be shown that condition (7) becomes true if a feasible, but non-optimal point is approached. Enforcing decrease in the objective function by (8) then prevents that the method converges to such a point. In accordance with previous publications on filter methods (e.g. [6, 8]) we may call a trial step size $\alpha_{k,l}$ for which (7) holds, an "f-step size."

In order to prevent the method from cycling, the algorithm maintains a "filter" $\mathcal{F}_k \subseteq \{(\theta, f) \in \mathbb{R}^2 : \theta \geq 0\}$, a set of (θ, f)-pairs that are "prohibited" for a trial point in iteration k. During the line search, a trial point $x_k(\alpha_{k,l})$ is rejected, if it is not acceptable to the current filter, i.e. if $(\theta(x_k(\alpha_{k,l})), f(x_k(\alpha_{k,l}))) \in \mathcal{F}_k$. At the beginning of the optimization, the filter is initialized to

$$\mathcal{F}_0 := \{(\theta, f) \in \mathbb{R}^2 : \theta \geq \theta^{\max}\}. \tag{9}$$

Later, the filter is augmented for a new iteration using the update formula

$$\mathcal{F}_{k+1} := \mathcal{F}_k \cup \Big\{(\theta, f) \in \mathbb{R}^2 : \theta \geq (1-\gamma_\theta)\theta(x_k) \quad \text{and} \quad f \geq f(x_k) - \gamma_f \theta(x_k)\Big\}, \tag{10}$$

if the accepted trial step size does not satisfy the switching condition (7). In this way, the iterates cannot return back into the neighborhood of x_k. On the other hand, if (7) (and therefore also (8)) holds for the accepted step size, the filter remains unchanged. Because such an iteration guarantees progress in the objective function, we may call it an "f-type iteration."

Finally, in some cases it is not possible to find a trial step size $\alpha_{k,l}$ that satisfies the above criteria. Using linear models of the involved functions, we assume to be in this situation, if $\alpha_{k,l}$ becomes smaller than

$$\alpha_k^{\min} := \gamma_\alpha \cdot \begin{cases} \min\left\{\gamma_\theta, \frac{\gamma_f \theta(x_k)}{-g_k^T d_k}, \frac{\delta[\theta(x_k)]^{s_\theta}}{[-g_k^T d_k]^{s_f}}\right\} & \text{if } g_k^T d_k < 0 \\ \gamma_\theta & \text{otherwise,} \end{cases} \tag{11}$$

[1] For the global convergence analysis in [14] it is sufficient if the constant s_f satisfies $s_f \geq 1$. However, for the proofs in this paper it has to satisfy a tighter condition, so that the relationship (26) below holds.

with a "safety factor" $\gamma_\alpha \in (0,1]$. If the backtracking line search encounters a trial step size with $\alpha_{k,l} \leq \alpha_k^{\min}$, the algorithm reverts to a *feasibility restoration phase*. Here, the algorithm tries to find a new iterate x_{k+1} that is acceptable to the current filter and for which (6) holds, by reducing the constraint violation with some iterative method. Note, that a suitable restoration phase algorithm might not be able to produce a new iterate for the filter line search method and instead converge to a local minimizer of the constraint violation, indicating to the user that the problem seems (at least locally) infeasible.

3 Second Order Correction Steps

It has been noted by Fletcher and Leyffer [7] that the filter approach, similar to a penalty function approach, can suffer from the Maratos effect. Here, a full Newton (or Newton-type) step increases *both* the objective function and the constraint violation, even arbitrarily close to a local solution of the NLP (1). As a consequence, the filter line search procedure rejects the full Newton step and only accept small fractions of the step. This can result in poor local convergence behavior. As a remedy, Fletcher and Leyffer propose to improve the search direction by means of a second order correction.

A second order correction step $d_k^{\rm soc}$ aims to reduce infeasibility by applying an additional Newton-type step for the constraints at the point $x_k + d_k$. There is a wide range of options to compute such a step. Here, we assume that it is obtained from the solution of the linear system

$$\begin{bmatrix} H_k^{\rm soc} & A_k^{\rm soc} \\ (A_k^{\rm soc})^T & 0 \end{bmatrix} \begin{pmatrix} d_k^{\rm soc} \\ \lambda_k^{\rm soc} \end{pmatrix} = - \begin{pmatrix} g_k^{\rm soc} \\ c(x_k + d_k) + c_k^{\rm soc} \end{pmatrix}, \tag{12}$$

where $H_k^{\rm soc}$ is a symmetric $n \times n$ matrix, $A_k^{\rm soc} \in \mathbb{R}^{n\times m}$, $g_k^{\rm soc} \in \mathbb{R}^n$, and $c_k^{\rm soc} \in \mathbb{R}^m$. Second order correction steps of the form (12) are discussed by Conn, Gould, and Toint in [3, Section 15.3.2.3]. We assume that $H_k^{\rm soc}$ is uniformly positive definite on the null space of $(A_k^{\rm soc})^T$, and that in a neighborhood of a second order sufficient solution we have

$$g_k^{\rm soc} = o(\|d_k\|), \qquad A_k - A_k^{\rm soc} = O(\|d_k\|), \qquad c_k^{\rm soc} = o(\|d_k\|^2). \tag{13}$$

In [3], the analysis is made for the particular choices $c_k^{\rm soc} = 0$, $A_k^{\rm soc} = A(x_k + p_k)$ for some $p_k = O(\|d_k\|)$, and $H_k = \nabla_{xx}^2 \mathcal{L}_\mu(x_k, \lambda_k)$ in (3) for multiplier estimates λ_k. However, the careful reader will be able to verify that the cited results from [3] still hold as long as

$$(W_k - H_k) d_k = o(\|d_k\|), \tag{14}$$

if x_k converges to a second order sufficient solution x_* of the NLP with corresponding multipliers λ_* (see Assumption (L2) below), where

$$W_k = \nabla_{xx}^2 \mathcal{L}(x_k, \lambda_*) \overset{(4)}{=} \nabla^2 f(x_k) + \sum_{i=1}^m (\lambda_*)^{(i)} \nabla^2 c^{(i)}(x_k). \tag{15}$$

We note that if we choose $H_k := \nabla_{xx}^2 \mathcal{L}_\mu(x_k, \lambda_k)$ where the sequence of multiplier estimates $\{\lambda_k\}$ is generated using λ_k^+ from (3) (e.g. by setting $\lambda_{k+1} := \lambda_k^+$), then (14) holds if x_k converges to a second order sufficient local solution x_* satisfying Assumption (L2) below.

Possible choices for the quantities in the computation of the second order correction step in (12) that satisfy (13) are the following.

(SOC-1) $H_k^{\rm soc} = I$, $g_k^{\rm soc} = 0$, $c_k^{\rm soc} = 0$, and $A_k^{\rm soc} = A_k$ or $A_k^{\rm soc} = A(x_k + d_k)$; this corresponds to a least-squares step for the constraints.

(SOC-2) $H_k^{\rm soc} = H_k$, $g_k^{\rm soc} = 0$, $c_k^{\rm soc} = 0$, and $A_k^{\rm soc} = A_k$; this option is inexpensive since it allows to re-use the factorization of the linear system (3).

(SOC-3) $H_k^{\rm soc}$ being the Hessian approximation corresponding to $x_k + d_k$, $g_k^{\rm soc} = g(x_k + d_k) + A(x_k + d_k)\lambda_k^+$, $c_k^{\rm soc} = 0$, and $A_k^{\rm soc} = A(x_k + d_k)$; this step corresponds to the step in the next iteration, supposing that $x_k + d_k$ has been accepted. In this sense, this choice has the flavor of the watchdog technique [2].

(SOC-4) If $d_k^{\rm soc}$ is a second order correction step, and $\bar{d}_k^{\rm soc}$ is an additional second order correction step (i.e. with "$c(x_k + d_k)$" replaced by "$c(x_k + d_k + d_k^{\rm soc})$" in (12)), then $d_k^{\rm soc} + \bar{d}_k^{\rm soc}$ can be understood as a single second order correction step for d_k (in that case with $c_k^{\rm soc} \neq 0$). Similarly, several consecutive correction steps can be considered as a single one.

It is easy to show that for the combined step $d_k + d_k^{\rm soc}$ we have $c(x_k + d_k + d_k^{\rm soc}) = o(\|d_k\|^2)$, (see (21b) below). As a consequence, the combined step has a better chance to be accepted by the filter method than the original step d_k if x_k is close to a local solution. In order to overcome the Maratos effect, we modify the filter line search procedure outlined in Section 2, so that a second order correction step is tried whenever the full step has not been accepted. As we see in Section 4, this indeed enables the algorithm to accept full steps close to a second order sufficient solution of (1), so that fast local convergence is achieved.

We now formally state the line search filter algorithm from [14] with the modification to include second order correction steps.

Algorithm I

Given: Starting point x_0; constants $\theta_{\max} \in (\theta(x_0), \infty]$; $\gamma_\theta, \gamma_f \in (0,1)$; $\delta > 0$; $\gamma_\alpha \in (0,1]$; $s_\theta > 1$; $s_f > 2s_\theta$; $\eta_f \in (0, \frac{1}{2})$; $0 < \tau_1 \leq \tau_2 < 1$.

1. *Initialize.* Initialize the filter (using (9)) and the iteration counter $k \leftarrow 0$.

2. *Check convergence.* Stop, if x_k is a local solution (or at least stationary point) of the NLP (1), i.e. if it satisfies the KKT conditions (2) for some $\lambda \in \mathbb{R}^m$.

3. *Compute search direction.* Compute the search direction d_k from the linear system (3). If this system is detected to be too ill-conditioned or singular, go to feasibility restoration phase in Step 8.

4. *Backtracking line search.*

   4.1. *Initialize line search.* Set $\alpha_{k,0} = 1$ and $l \leftarrow 0$.

   4.2. *Compute new trial point.* If the trial step size becomes too small, i.e. $\alpha_{k,l} < \alpha_k^{\min}$ with $\alpha_k^{\min}$ defined by (11), go to the feasibility restoration phase in Step 8. Otherwise, compute the new trial point $x_k(\alpha_{k,l}) := x_k + \alpha_{k,l} d_k$.

   4.3. *Check acceptability to the filter.* If $(\theta(x_k(\alpha_{k,l})), f(x_k(\alpha_{k,l})) \in \mathcal{F}_k$, reject the trial step size and go to Step 4.5.

   4.4. *Check sufficient decrease with respect to current iterate.*

      4.4.1. *Case I: $\alpha_{k,l}$ is an f-step-size (i.e. (7) holds):* If the Armijo condition (8) for the objective function holds, accept the trial step $x_{k+1} := x_k(\alpha_{k,l})$ and go to Step 5. Otherwise, go to Step 4.5.

4.4.2. *Case II: $\alpha_{k,l}$ is not an f-step-size (i.e. (7) is not satisfied):* If (6) holds, accept the trial step $x_{k+1} := x_k(\alpha_{k,l})$ and go to Step 5. Otherwise, go to Step 4.5.

4.5. *Compute second order correction step.* If $l \neq 0$, go to step 4.8. Otherwise, solve the linear system (12) to obtain the second order correction step d_k^{soc} and define

$$\bar{x}_{k+1} := x_k + d_k + d_k^{\mathrm{soc}}.$$

4.6. *Check acceptability to the filter.* If $\bar{x}_{k+1} \in \mathcal{F}_k$, reject the second order correction step and go to Step 4.8.

4.7. *Check sufficient decrease with respect to current iterate.*

4.7.1. *Case I: The switching condition (7) holds (for $\alpha_{k,0}$ and d_k):* If the Armijo condition for the objective function,

$$f(\bar{x}_{k+1}) \leq f(x_k) + \eta_f \, g_k^T d_k, \tag{16}$$

holds, accept $x_{k+1} := \bar{x}_{k+1}$ and go to Step 5. Otherwise, go to Step 4.8.

4.7.2. *Case II: The switching condition (7) is not satisfied:* If

$$\theta(\bar{x}_{k+1}) \leq (1-\gamma_\theta)\theta(x_k) \tag{17a}$$

$$\text{or} \quad f(\bar{x}_{k+1}) \leq f(x_k) - \gamma_f \theta(x_k) \tag{17b}$$

hold, accept $x_{k+1} := \bar{x}_{k+1}$ and go to Step 5. Otherwise, go to Step 4.8.

4.8. *Choose new trial step size.* Choose $\alpha_{k,l+1} \in [\tau_1 \alpha_{k,l}, \tau_2 \alpha_{k,l}]$, set $l \leftarrow l+1$, and go back to Step 4.2.

5. *Accept trial point.* Set $\alpha_k := \alpha_{k,l}$.

6. *Augment filter if necessary.* If k is not an f-type iteration (i.e., (7) does not hold for α_k), augment the filter using (10); otherwise leave the filter unchanged, i.e. set $\mathcal{F}_{k+1} := \mathcal{F}_k$.

7. *Continue with next iteration.* Increase the iteration counter $k \leftarrow k+1$ and go back to Step 2.

8. *Feasibility restoration phase.* Compute a new iterate x_{k+1} by decreasing the infeasibility measure θ, so that x_{k+1} satisfies the sufficient decrease conditions (6) and is acceptable to the filter, i.e. $(\theta(x_{k+1}), f(x_{k+1})) \notin \mathcal{F}_k$. Augment the filter using (10) (for x_k) and continue with the regular iteration in Step 7.

It can be verified easily that this modification of Algorithm I in the companion paper [14] does not affect the global convergence properties proved in [14].

4 Local Convergence Analysis

We start the analysis by stating the necessary assumptions.

Assumptions L. *Assume that the algorithm generates an infinite sequence $\{x_k\}$ of iterates that converges to a local solution x_* of the NLP (1), and that the following holds.*

(L1) *The functions f and c are twice continuously differentiable in a neighborhood of x_*.*

(L2) *x_* satisfies the following sufficient second order optimality conditions.*

- *There exists $\lambda_* \in \mathbb{R}^m$ so that the KKT conditions (2) are satisfied for (x_*, λ_*);*
- *the constraint Jacobian $A(x_*)^T$ has full rank; and*
- *the Hessian of the Lagrangian $W_* = \nabla^2_{xx}\mathcal{L}(x_*, \lambda_*)$ is positive definite on the null space of $A(x_*)^T$.*

(L3) *In (3), H_k is uniformly positive definite on the null space of $(A_k)^T$, as well as bounded.*

(L4) *In (12), H_k^{soc} is uniformly positive definite on the null space of $(A_k^{\text{soc}})^T$, and (13) holds.*

(L5) *The matrices H_k in (3) satisfy (14).*

(L6) *There exists a constant $\theta_{\text{inc}} > 0$, so that the algorithm does not switch in Step 3 to the restoration phase if $\theta(x_k) \le \theta_{\text{inc}}$.*

The assumption "$\lim_k x_k = x_*$" is discussed in Remark 8 in the companion paper [14]. It is shown that if a particular restoration phase algorithm (based on Newton steps for the KKT conditions) is used in the neighborhood of a solution x_* satisfying (L2), then the iterates of the overall filter line search algorithm are attracted to x_* so that $x_k \to x_*$ follows. Assumption (L5) is reminiscent of the Dennis-Moré characterization of superlinear convergence [4], but it is stronger than the one necessary for superlinear convergence [1] which requires only that $Z_k^T(W_k - H_k)d_k = o(\|d_k\|)$, where Z_k is a null space matrix for A_k^T. However, if multiplier estimates λ_k based on λ_k^+ from (3) and exact second derivatives are used to obtain H_k close to x_*, i.e. if

$$H_k = \nabla^2_{xx}\mathcal{L}(x_k, \lambda_k), \tag{18}$$

then Assumptions (L3) and (L5) are satisfied, since $H_k \to W_*$ in that case. Finally, the algorithm allows to revert to the restoration phase in Step 3. This option exists so that the overall globally convergent line search method can handle infeasible points at which the constraint gradients are linearly dependent (see [14] for details). Therefore, Assumption (L6) is introduced as a formality to guarantee that the algorithm does not switch in arbitrary iterations to the restoration phase close to feasible points. In light of Assumption (L2), it is easy to see that the iteration matrix in (3) is nonsingular close to x_*, if H_k is chosen to be close to W_* (e.g. by (18)), so that there is not need to revert to the restoration phase in Step 3 close to x_*, and Assumption (L6) is satisfied.

The above assumptions imply Assumptions G in the companion paper [14] in a neighborhood of the solution. Therefore, Lemma 1 from [14] remains valid close to x_*, which states that d_k and λ_k^+ from (3) are uniformly bounded. Furthermore, as can be verified easily, the proof of Lemma 4 in [14] holds using Assumptions (L3) and (L6), so that

$$\theta(x_k) = 0 \quad \Longrightarrow \quad g_k^T d_k < 0 \qquad \text{and} \tag{19}$$

$$\Theta_k := \min\{\theta : (\theta, f) \in \mathcal{F}_k\} > 0 \tag{20}$$

for all k.

First we summarize some preliminary results.

Lemma 1 *Suppose Assumptions L hold. Then there exists a neighborhood U_1 of x_*, so that for all $x_k \in U_1$ we have*

$$d_k^{\text{soc}} = o(\|d_k\|) \tag{21a}$$

$$c(x_k + d_k + d_k^{\text{soc}}) = o(\|d_k\|^2) \tag{21b}$$

Proof. From continuity, condition (13), and full rank of A_*^T, as well as Assumption (L4), we have that the matrix in (12) has a uniformly bounded inverse in the neighborhood of x_*. Hence, since the right hand side is $o(\|d_k\|)$, claim (21a) follows. Furthermore, from

$$\begin{aligned} c(x_k + d_k + d_k^{\mathrm{soc}}) &= c(x_k + d_k) + A(x_k + d_k)^T d_k^{\mathrm{soc}} + O(\|d_k^{\mathrm{soc}}\|^2) \\ &\overset{(12)}{=} -c_k^{\mathrm{soc}} - (A_k^{\mathrm{soc}})^T d_k^{\mathrm{soc}} + (A_k + O(\|d_k\|))^T d_k^{\mathrm{soc}} \\ &\qquad + O(\|d_k^{\mathrm{soc}}\|^2) \\ &\overset{(13)}{=} o(\|d_k\|^2) + O(\|d_k\|\|d_k^{\mathrm{soc}}\|) + O(\|d_k^{\mathrm{soc}}\|^2) \\ &\overset{(21a)}{=} o(\|d_k\|^2) \end{aligned}$$

for x_k close to x_* the claim (21b) follows. $\square$

In order to prove our local convergence result we make use of two results established in [3] regarding the effect of second order correction steps on the exact penalty function

$$\phi_\rho(x) = f(x) + \rho\, \theta(x). \tag{22}$$

Note, that we employ the exact penalty function only as a technical device, but the algorithm never refers to it. We also use the following model of the penalty function

$$q_\rho(x_k, d) = f(x_k) + g_k^T d + \frac{1}{2} d^T H_k d + \rho \left\| A_k^T d + c_k \right\|. \tag{23}$$

The first result follows from Theorem 15.3.7 in [3].

Lemma 2 *Suppose Assumptions L hold. Let ϕ_ρ be the exact penalty function (22) and q_ρ defined by (23) with $\rho > \|\lambda_*\|_D$, where $\|\cdot\|_D$ is the dual norm to $\|\cdot\|$. Then,*

$$\lim_{k\to\infty} \frac{\phi_\rho(x_k) - \phi_\rho(x_k + d_k + d_k^{\mathrm{soc}})}{q_\rho(x_k, 0) - q_\rho(x_k, d_k)} = 1. \tag{24}$$

The next result follows from Theorem 15.3.2 in [3].

Lemma 3 *Suppose Assumptions L hold. Let (d_k, λ_k^+) be a solution of the linear system (3), and let $\rho > \|\lambda_k^+\|_D$. Then*

$$q_\rho(x_k, 0) - q_\rho(x_k, d_k) \geq 0. \tag{25}$$

The next lemma shows that in a neighborhood of x_*, Step 4.7.1 of Algorithm I is successful if the combined step $d_k + d_k^{\mathrm{soc}}$ is an f-type step.

Lemma 4 *Suppose Assumptions L hold. Then there exists a neighborhood $U_2 \subseteq U_1$ of x_* so that whenever (7) holds for $\alpha_{k,l} = 1$, the Armijo condition (16) is satisfied.*

Proof. Choose U_1 to be the neighborhood from Lemma 1. It then follows that for $x_k \in U_1$ satisfying (7) that

$$\theta(x_k) < \delta^{-\frac{1}{s_\theta}} [-g_k^T d_k]^{\frac{s_f}{s_\theta}} = O(\|d_k\|^{\frac{s_f}{s_\theta}}) = o(\|d_k\|^2), \tag{26}$$

since $\frac{s_f}{s_\theta} > 2$ and g_k is uniformly bounded in U_1.

Since $\eta_f < \frac{1}{2}$, Lemma 2 and (25) imply that there exists $K \in \mathbb{N}$ such that for all $k \geq K$ we have for some constant $\rho > 0$ with $\rho > \|\lambda_k^+\|_D$ independent of k that

$$\phi_\rho(x_k) - \phi_\rho(x_k + d_k + d_k^{\mathrm{soc}}) \geq \left(\frac{1}{2} + \eta_f\right) (q_\rho(x_k, 0) - q_\rho(x_k, d_k)). \tag{27}$$

We then have

$$
\begin{aligned}
& f(x_k) - f(x_k + d_k + d_k^{\text{soc}}) \\
\overset{(22)}{=} \quad & \phi_\rho(x_k) - \phi_\rho(x_k + d_k + d_k^{\text{soc}}) - \rho\left(\theta(x_k) - \theta(x_k + d_k + d_k^{\text{soc}})\right) \\
\overset{(27),(21b),(26)}{\geq} \quad & \left(\frac{1}{2} + \eta_f\right)\left(q_\rho(x_k, 0) - q_\rho(x_k, d_k)\right) + o(\|d_k\|^2) \\
\overset{(23),(26),(3)}{=} \quad & -\left(\frac{1}{2} + \eta_f\right)\left(g_k^T d_k + \frac{1}{2} d_k^T H_k d_k\right) + o(\|d_k\|^2). \qquad (28)
\end{aligned}
$$

Before continuing, we recall the step decomposition from the companion paper [14]

$$
\begin{aligned}
d_k &= q_k + p_k, & (29a)\\
q_k &:= Y_k \bar{q}_k \quad \text{and} \quad p_k := Z_k \bar{p}_k, & (29b)\\
\bar{q}_k &:= -\left[A_k^T Y_k\right]^{-1} c_k & (29c)\\
\bar{p}_k &:= -\left[Z_k^T H_k Z_k\right]^{-1} Z_k^T (g_k + H_k q_k) & (29d)
\end{aligned}
$$

where $Y_k \in \mathbb{R}^{n \times m}$ and $Z_k \in \mathbb{R}^{n \times (n-m)}$ are matrices so that the columns of $[Y_k \ Z_k]$ form an orthonormal basis of $\mathbb{R}^n$, and the columns of Z_k are a basis of the null space of A_k^T. Since Assumptions L guarantee that the quantities (29), as well as λ_k^+, are bounded for k sufficiently large, we can conclude

$$
\begin{aligned}
& f(x_k) + \eta_f g_k^T d_k - f(x_k + d_k + d_k^{\text{soc}}) \\
\overset{(28)}{\geq} \quad & -\frac{1}{2} g_k^T d_k - \left(\frac{1}{4} + \frac{\eta_f}{2}\right) d_k^T H_k d_k + o(\|d_k\|^2) \\
\overset{(3)}{=} \quad & \frac{1}{2}\left(d_k^T H_k d_k + d_k^T A_k \lambda_k^+\right) - \left(\frac{1}{4} + \frac{\eta_f}{2}\right) d_k^T H_k d_k + o(\|d_k\|^2) \\
\overset{(3)}{=} \quad & \left(\frac{1}{4} - \frac{\eta_f}{2}\right) d_k^T H_k d_k - \frac{1}{2} c(x_k)^T \lambda_k^+ + o(\|d_k\|^2) \\
\overset{(26)}{=} \quad & \left(\frac{1}{4} - \frac{\eta_f}{2}\right) d_k^T H_k d_k + o(\|d_k\|^2) \\
\overset{(29)}{=} \quad & \left(\frac{1}{4} - \frac{\eta_f}{2}\right) \bar{p}_k^T Z_k^T H_k Z_k \bar{p}_k + O(\|q_k\|) + o(\|d_k\|^2). \qquad (30)
\end{aligned}
$$

Finally, using repeatedly the orthonormality of $[Y_k \ Z_k]$, we have

$$
\begin{aligned}
q_k \quad &= \quad O(\bar{q}_k) \overset{(29c)}{=} O(\theta(x_k)) \overset{(26)}{=} o(\|d_k\|^2) \\
&\overset{(29a)}{=} \quad o(p_k^T p_k + q_k^T q_k) \overset{(29b)}{=} o(\|\bar{p}_k\|^2) + o(\|q_k\|^2)
\end{aligned}
$$

and therefore $q_k = o(\|\bar{p}_k\|^2)$, as well as

$$
d_k \overset{(29a)}{=} O(\|q_k\|) + O(\|p_k\|) \overset{(29b)}{=} o(\|\bar{p}_k\|^2) + O(\|\bar{p}_k\|) = O(\|\bar{p}_k\|).
$$

Since $\bar{p}_k \to 0$ as $x_k \to x_*$, (16) is then implied by (30), Assumption (L3) and $\eta_f < \frac{1}{2}$, if x_k is sufficiently close to x_*. $\square$

It remains to show that also the filter and the sufficient reduction criterion (6) do not interfere with the acceptance of full steps close to x_*. The following technical lemmas address this issue and prepare the proof of the main local convergence theorem.

Lemma 5 *Suppose Assumptions L hold. Then there exists a neighborhood $U_3 \subseteq U_2$ (with U_2 from Lemma 4) and constants $\rho_1, \rho_2, \rho_3 > 0$ with*

$$\begin{aligned} \rho_3 &= (1-\gamma_\theta)\rho_2 - \gamma_f & \text{(31a)} \\ 2\gamma_\theta \rho_2 &< (1+\gamma_\theta)(\rho_2 - \rho_1) - 2\gamma_f & \text{(31b)} \\ 2\rho_3 &\geq (1+\gamma_\theta)\rho_1 + (1-\gamma_\theta)\rho_2, & \text{(31c)} \end{aligned}$$

so that for all $x_k \in U_3$ we have $\|\lambda_k^+\|_D < \rho_i$ for $i = 1,2,3$. Furthermore, for all $x_k \in U_3$ we have

$$\phi_{\rho_i}(x_k) - \phi_{\rho_i}(x_k + d_k + \bar{d}_k^{\text{soc}}) \geq \frac{1+\gamma_\theta}{2}\left(q_{\rho_i}(x_k, 0) - q_{\rho_i}(x_k, d_k)\right) \overset{(25)}{\geq} 0 \tag{32}$$

for $i = 2,3$ and all choices

$$\begin{aligned} \bar{d}_k^{\text{soc}} &= d_k^{\text{soc}}, & \text{(33a)} \\ \bar{d}_k^{\text{soc}} &= \sigma_k d_k^{\text{soc}} + d_{k+1} + \sigma_{k+1} d_{k+1}^{\text{soc}}, & \text{(33b)} \\ \bar{d}_k^{\text{soc}} &= \sigma_k d_k^{\text{soc}} + d_{k+1} + \sigma_{k+1} d_{k+1}^{\text{soc}} + d_{k+2} + \sigma_{k+2} d_{k+2}^{\text{soc}}, & \text{(33c)} \\ \textit{or} \qquad \bar{d}_k^{\text{soc}} &= \sigma_k d_k^{\text{soc}} + d_{k+1} + \sigma_{k+1} d_{k+1}^{\text{soc}} + d_{k+2} + \sigma_{k+2} d_{k+2}^{\text{soc}} \\ &\quad + d_{k+3} + \sigma_{k+3} d_{k+3}^{\text{soc}}, & \text{(33d)} \end{aligned}$$

with $\sigma_k, \sigma_{k+1}, \sigma_{k+2}, \sigma_{k+3} \in \{0,1\}$, as long as $x_{l+1} = x_l + d_l + \sigma_l d_k^{\text{soc}}$ for $l \in \{k, \ldots, k+j\}$ with $j \in \{-1, 0, 1, 2\}$, respectively.

Proof. Since λ_k^+ is uniformly bounded for all k with $x_k \in U_2$, we can find $\rho_1 > \|\lambda_*\|_D$ with

$$\rho_1 > \|\lambda_k^+\|_D \tag{34}$$

for all k with $x_k \in U_2$. Defining now

$$\rho_2 := \frac{1+\gamma_\theta}{1-\gamma_\theta}\rho_1 + \frac{3\gamma_f}{1-\gamma_\theta}$$

and ρ_3 by (31a), it is then easy to verify that $\rho_2, \rho_3 \geq \rho_1 > \|\lambda_k^+\|_D$ and that (31b) and (31c) hold. Since $(1+\gamma_\theta) < 2$, Lemma 2 implies that there exists a neighborhood $U_3 \subseteq U_2$ of x_*, so that (32) holds for $x_k \in U_3$, since according to the second order correction step choices (SOC-3) and (SOC-4) in Section 3 all options for $\bar{d}_k^{\text{soc}}$ in (33) can be understood as second order correction steps to d_k. □

Before proceeding we give a short graphical motivation of the remainder of the proof and introduce some more notation. Let U_3 and ρ_i be the neighborhood and constants from Lemma 5. Since $\lim_k x_k = x_*$, we can find $K_1 \in \mathbb{N}$ so that $x_k \in U_3$ for all $k \geq K_1$. In Figure 1 we see the (θ, f) half-plane with the current filter $\mathcal{F}_{K_1}$. Let us now define the level set

$$M := \{x \in U_3 : \phi_{\rho_3}(x) \leq \phi_{\rho_3}(x_*) + \kappa\}, \tag{35}$$

where $\kappa > 0$ is chosen so that for all $x \in M$ we have $(\theta(x), f(x)) \notin \mathcal{F}_{K_1}$. This is possible, since $\Theta_{K_1} > 0$ from (20), and since $\max\{\theta(x) : x \in M\}$ converges to zero as $\kappa \to 0$, because x_* is a strict local minimizer of ϕ_{ρ_3} [9]. Obviously, $x_* \in M$.

In Figure 1, $\mathcal{M}$ and $\mathcal{U}_3$ are the images of M and U_3 in the (θ, f) half-plane. Let K_2 now be the first iteration $K_2 \geq K_1$ with $x_{K_2} \in M$. This means, that no iterate after K_1 and before K_2 is in M, and therefore also the filter $\mathcal{F}_{K_2}$ overlaps with $\mathcal{M}$ by at most a small area whose size is governed by the parameters γ_f and γ_θ. The (θ, f)-pairs with constant value of the exact

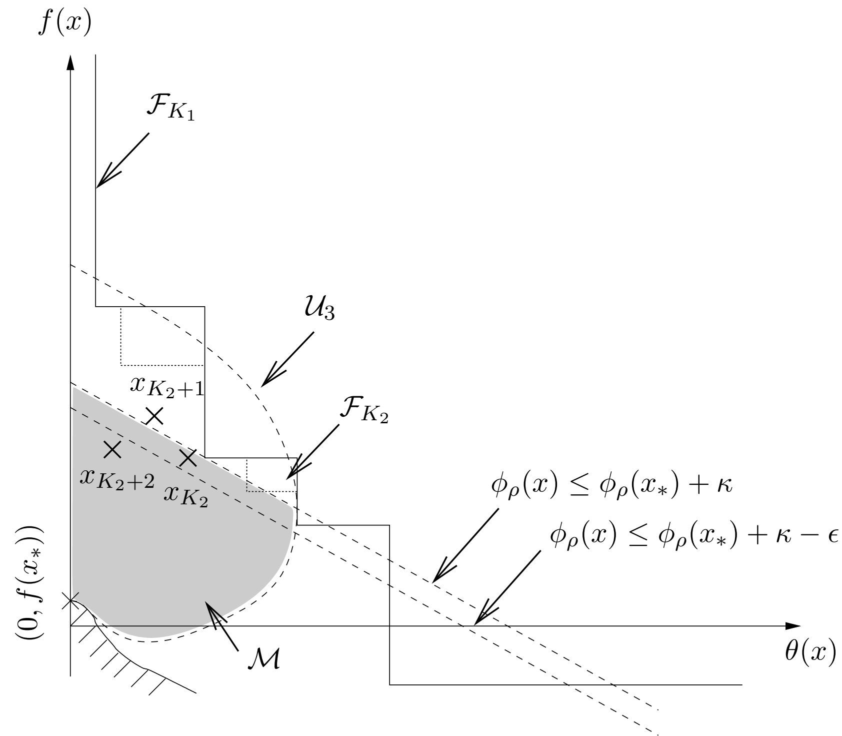

Figure 1: Basic idea of proof

penalty function (22) correspond to dashed lines in the diagram, the slope of which is determined by the penalty parameter ρ. The main trick of the proof is to use these dashed lines as frontiers approaching $(0, f(x_*))$, so that the filter always lies to the upper right side of these lines (except for small overlaps coming from (10) in the filter update rule), and at least every other iterate (with or without second order correction step) lies on the lower left side of these lines (see (32)). For technical reasons we have to consider three of those frontiers corresponding to different values of the penalty parameter, in order to deal with sufficient progress with respect to the old filter entries, the current iterate (6), and new filter entries.

We denote the set of iteration indices k, in which the filter is augmented, by $\mathcal{A} \subseteq \mathbb{N}$; i.e.

$$\mathcal{F}_k \subsetneqq \mathcal{F}_{k+1} \qquad \Longleftrightarrow \qquad k \in \mathcal{A}.$$

Also, we define for $k \in \mathbb{N}$ the filter building blocks

$$\mathcal{G}_k := \Big\{(\theta, f) : \theta \ge (1-\gamma_\theta)\theta(x_k) \quad \text{and} \quad f \ge f(x_k) - \gamma_f \theta(x_k)\Big\}$$

and index sets $I_{k_1}^{k_2} := \{l = k_1, \dots, k_2 - 1 : l \in \mathcal{A}\}$ for $k_1 \le k_2$. Then it follows from the filter update rule (10) and the definition of $\mathcal{A}$ that for $k_1 \le k_2$

$$\mathcal{F}_{k_2} = \mathcal{F}_{k_1} \cup \bigcup_{l \in I_{k_1}^{k_2}} \mathcal{G}_l. \tag{36}$$

Also note, that $l \in I_{k_1}^{k_2} \subseteq \mathcal{A}$ implies $\theta(x_l) > 0$. Otherwise, we would have from (19) that $g_k^T d_k < 0$, so that (7) holds for all trial step sizes α, and the step must have been accepted in Step 4.4.1 or Step 4.7.1, hence satisfying (8) or (16). This would contradict the filter update condition in Step 6.

The last lemma enables us to show in the main theorem of this section that, once the iterates have reached the level set M, the full step is always acceptable to the current filter.

Lemma 6 *Suppose Assumptions L hold and let $l \geq K_1$ with $\theta(x_l) > 0$. Then the following statements holds for a given $x \in \mathbb{R}^n$.*

$$\left.\begin{array}{l} \text{If } \phi_{\rho_2}(x_l) - \phi_{\rho_2}(x) \geq \frac{1+\gamma_\theta}{2}\left(q_{\rho_2}(x_l,0) - q_{\rho_2}(x_l,d_l)\right), \\ \text{then } (\theta(x), f(x)) \notin \mathcal{G}_l. \end{array}\right\} \tag{37}$$

$$\left.\begin{array}{l} \text{If } x \in M \text{ and } \phi_{\rho_2}(x_{K_2}) - \phi_{\rho_2}(x) \geq \frac{1+\gamma_\theta}{2}\left(q_{\rho_2}(x_{K_2},0) - q_{\rho_2}(x_{K_2},d_{K_2})\right), \\ \text{then } (\theta(x), f(x)) \notin \mathcal{F}_{K_2}. \end{array}\right\} \tag{38}$$

Proof. To (37): Since $\rho_1 > \|\lambda_l^+\|_D$ we have from Lemma 3 that $q_{\rho_1}(x_l,0) - q_{\rho_1}(x_l,d_l) \geq 0$. Hence, using the definition (23) for q_ρ, as well as $A_l^T d_l + c_l = 0$ (from (3)), we obtain

$$\begin{aligned} \phi_{\rho_2}(x_l) - \phi_{\rho_2}(x) &\geq \frac{1+\gamma_\theta}{2}\left(q_{\rho_2}(x_l,0) - q_{\rho_2}(x_l,d_l)\right) \\ &= \frac{1+\gamma_\theta}{2}\left(q_{\rho_1}(x_l,0) - q_{\rho_1}(x_l,d_l) + (\rho_2-\rho_1)\theta(x_l)\right) \\ &\overset{(25)}{\geq} \frac{1+\gamma_\theta}{2}(\rho_2-\rho_1)\theta(x_l). \end{aligned} \tag{39}$$

If $f(x) < f(x_l) - \gamma_f\theta(x_l)$, the claim follows immediately. Otherwise, suppose that $f(x) \geq f(x_l) - \gamma_f\theta(x_l)$. In that case, we have together with $\theta(x_l) > 0$ that

$$\begin{aligned} \theta(x_l) - \theta(x) &\overset{(22),(39)}{\geq} \frac{1+\gamma_\theta}{2\rho_2}(\rho_2-\rho_1)\theta(x_l) + \frac{1}{\rho_2}\left(f(x) - f(x_l)\right) \\ &\geq \frac{1+\gamma_\theta}{2\rho_2}(\rho_2-\rho_1)\theta(x_l) - \frac{\gamma_f}{\rho_2}\theta(x_l) \\ &\overset{(31b)}{>} \gamma_\theta\theta(x_l), \end{aligned}$$

so that $(\theta(x), f(x)) \notin \mathcal{G}_l$.

To (38): Since $x \in M$, it follows by the choice of κ in (35) that $(\theta(x), f(x)) \notin \mathcal{F}_{K_1}$. Thus, according to (36) it remains to show that for all $l \in I_{K_1}^{K_2}$ we have $(\theta(x), f(x)) \notin \mathcal{G}_l$. Choose $l \in I_{K_1}^{K_2}$. As in (39) we can show that

$$\phi_{\rho_2}(x_{K_2}) - \phi_{\rho_2}(x) \geq \frac{1+\gamma_\theta}{2}(\rho_2-\rho_1)\theta(x_{K_2}). \tag{40}$$

Since $x \in M$ it follows from the definition of K_2 (as the first iterate after K_1 with $x_{K_2} \in M$) and the fact that $l < K_2$ that

$$\begin{aligned} \phi_{\rho_3}(x_l) &\overset{(35)}{>} \phi_{\rho_3}(x_{K_2}) \overset{(22)}{=} \phi_{\rho_2}(x_{K_2}) + (\rho_3-\rho_2)\theta(x_{K_2}) \\ &\overset{(40)}{\geq} \phi_{\rho_2}(x) + \left(\rho_3 - \frac{1+\gamma_\theta}{2}\rho_1 - \frac{1-\gamma_\theta}{2}\rho_2\right)\theta(x_{K_2}) \\ &\overset{(31c)}{\geq} \phi_{\rho_2}(x). \end{aligned} \tag{41}$$

If $f(x) < f(x_l) - \gamma_f \theta(x_l)$, we immediately have $(\theta(x), f(x)) \notin \mathcal{G}_l$. Otherwise we have $f(x) \geq f(x_l) - \gamma_f \theta(x_l)$ which yields

$$\begin{aligned} \theta(x) &\overset{(22),(41)}{<} \frac{1}{\rho_2}\left(f(x_l) + \rho_3 \theta(x_l) - f(x)\right) \\ &\leq \frac{\rho_3 + \gamma_f}{\rho_2} \theta(x_l) \\ &\overset{(31a)}{=} (1 - \gamma_\theta)\theta(x_l), \end{aligned}$$

so that $(\theta(x), f(x)) \notin \mathcal{G}_l$ which concludes the proof of (38). □

After these preparations we are finally able to show the main local convergence theorem.

Theorem 1 *Suppose Assumptions L hold. Then, for k sufficiently large, full steps of the form $x_{k+1} = x_k + d_k$ or $x_{k+1} = x_k + d_k + d_k^{\rm soc}$ are taken, and x_k converges to x_* superlinearly.*

Proof. Recall that $K_2 \geq K_1$ is the first iteration after K_1 with $x_{K_2} \in M \subseteq U_3$. We now show by induction that the following statements are true for $k \geq K_2 + 2$:

$$\begin{aligned} &(\mathrm{i}_k) && \phi_{\rho_i}(x_l) - \phi_{\rho_i}(x_k) \geq \frac{1+\gamma_\theta}{2}\left(q_{\rho_i}(x_l, 0) - q_{\rho_i}(x_l, d_l)\right) \\ & && \qquad\qquad \text{for } i \in \{2,3\} \text{ and } K_2 \leq l \leq k-2 \\ &(\mathrm{ii}_k) && x_k \in M \\ &(\mathrm{iii}_k) && x_k = x_{k-1} + d_{k-1} + \sigma_{k-1} d_{k-1}^{\rm soc} \qquad \text{with } \sigma_{k-1} \in \{0,1\}. \end{aligned}$$

We start by showing that these statements are true for $k = K_2 + 2$.

Suppose, the point $x_{K_2} + d_{K_2}$ is not accepted by the line search. In that case, define $\bar{x}_{K_2+1} := x_{K_2} + d_{K_2} + d_{K_2}^{\rm soc}$. Then, from (32) with $i = 3$, $k = K_2$, and (33a), we see from $x_{K_2} \in M$ and the definition of M that $\bar{x}_{K_2+1} \in M$. After applying (32) again with $i = 2$ it follows from (38) that $(\theta(\bar{x}_{K_2+1}), f(\bar{x}_{K_2+1})) \notin \mathcal{F}_{K_2}$, i.e. $\bar{x}_{K_2+1}$ is not rejected in Step 4.6. Furthermore, if the switching condition (7) holds, we see from Lemma 4 that the Armijo condition (16) with $k = K_2$ is satisfied for the point $\bar{x}_{K_2+1}$. In the other case, i.e. if (7) is violated (note that then (19) and (7) imply $\theta(x_{K_2}) > 0$), we see from (32) for $i = 2$, $k = K_2$, and (33a), together with (37) for $l = K_2$, that (17) holds. Hence, $\bar{x}_{K_2+1}$ is also not rejected in Step 4.7 and accepted as next iterate. Summarizing the discussion in this paragraph we can write $x_{K_2+1} = x_{K_2} + d_{K_2} + \sigma_{K_2} d_{K_2}^{\rm soc}$ with $\sigma_{K_2} \in \{0,1\}$.

Let us now consider iteration $K_2 + 1$. For $\sigma_{K_2+1} \in \{0,1\}$ we have from (32) for $k = K_2$ and (33b) that

$$\begin{aligned} & \phi_{\rho_i}(x_{K_2}) - \phi_{\rho_i}(x_{K_2+1} + d_{K_2+1} + \sigma_{K_2+1} d_{K_2+1}^{\rm soc}) \\ \geq\; & \frac{1+\gamma_\theta}{2}\left(q_{\rho_i}(x_{K_2}, 0) - q_{\rho_i}(x_{K_2}, d_{K_2})\right) \end{aligned} \tag{42}$$

for $i = 2, 3$, which yields

$$x_{K_2+1} + d_{K_2+1} + \sigma_{K_2+1} d_{K_2+1}^{\rm soc} \in M. \tag{43}$$

If $x_{K_2+1} + d_{K_2+1}$ is accepted as next iterate x_{K_2+2}, we immediately obtain from (42) and (43) that (i_{K_2+2})–(iii_{K_2+2}) hold. Otherwise, we consider the case $\sigma_{K_2+1} = 1$. From (42), (43), and (38) we have for $\bar{x}_{K_2+2} := x_{K_2+1} + d_{K_2+1} + d_{K_2+1}^{\rm soc}$ that $(\theta(\bar{x}_{K_2+2}), f(\bar{x}_{K_2+2})) \notin \mathcal{F}_{K_2}$. If $K_2 \notin I_{K_2}^{K_2+1}$ it immediately follows from (36) that $(\theta(\bar{x}_{K_2+2}), f(\bar{x}_{K_2+2})) \notin \mathcal{F}_{K_2+1}$. Otherwise, we have $\theta(x_{K_2}) > 0$. Then, (42) for $i = 2$ together with (37) implies $(\theta(\bar{x}_{K_2+2}), f(\bar{x}_{K_2+2})) \notin \mathcal{G}_{K_2}$, and hence with (36) we have $(\theta(\bar{x}_{K_2+2}), f(\bar{x}_{K_2+2})) \notin \mathcal{F}_{K_2+1}$, so that $\bar{x}_{K_2+2}$ is not rejected in Step 4.6. Arguing similarly as in the previous paragraph we can conclude that $\bar{x}_{K_2+2}$ is also not rejected in Step 4.7. Therefore, $x_{K_2+2} = \bar{x}_{K_2+2}$. Together with (42) and (43) this proves (i$_{K_2+2}$)–(iii$_{K_2+2}$) for the case $\sigma_{K_2+1} = 1$.

Now suppose that (i$_l$)–(iii$_l$) are true for all $K_2 + 2 \leq l \leq k$ with some $k \geq K_2 + 2$. If $x_k + d_k$ is accepted by the line search, define $\sigma_k := 0$, otherwise $\sigma_k := 1$. Set $\bar{x}_{k+1} := x_k + d_k + \sigma_k d_k^{\mathrm{soc}}$. From (32) for (33c) we then have for $i = 2, 3$

$$\phi_{\rho_i}(x_{k-1}) - \phi_{\rho_i}(\bar{x}_{k+1}) \geq \frac{1+\gamma_\theta}{2}\left(q_{\rho_i}(x_{k-1}, 0) - q_{\rho_i}(x_{k-1}, d_{k-1})\right) \geq 0. \tag{44}$$

Choose l with $K_2 \leq l < k - 1$ and consider two cases:

Case a): If $k = K_2 + 2$, then $l = K_2$, and it follows from (32) with (33d) that for $i = 2, 3$

$$\phi_{\rho_i}(x_l) - \phi_{\rho_i}(\bar{x}_{k+1}) \geq \frac{1+\gamma_\theta}{2}\left(q_{\rho_i}(x_l, 0) - q_{\rho_i}(x_l, d_l)\right) \geq 0. \tag{45}$$

Case b): If $k > K_2 + 2$, we have from (44) that $\phi_{\rho_i}(\bar{x}_{k+1}) \leq \phi_{\rho_i}(x_{k-1})$ and hence from (i$_{k-1}$) it follows that (45) also holds in this case.

In either case, (45) implies in particular that $\phi_{\rho_3}(\bar{x}_{k+1}) \leq \phi_{\rho_3}(x_{K_2})$, and since $x_{K_2} \in M$, we obtain

$$\bar{x}_{k+1} \in M. \tag{46}$$

If $x_k + d_k$ is accepted by the line search, (i$_{k+1}$)–(iii$_{k+1}$) follow from (45), (44) and (46). If $x_k + d_k$ is rejected, we see from (46), (45) for $i = 2$ and $l = K_2$, and (38) that $(\theta(\bar{x}_{k+1}), f(\bar{x}_{k+1})) \notin \mathcal{F}_{K_2}$. Furthermore, for $l \in I_{K_2}^k$ we have from (44) and (45) with (37) that $(\theta(\bar{x}_{k+1}), f(\bar{x}_{k+1})) \notin \mathcal{G}_l$, and hence from (36) that $\bar{x}_{k+1}$ is not rejected in Step 4.6. We can again show as before that $\bar{x}_{k+1}$ is not rejected in Step 4.7, so that $x_{k+1} = \bar{x}_{k+1}$ which implies (i$_{k+1}$)–(iii$_{k+1}$).

That $\{x_k\}$ converges to x_* with a superlinear rate follows from (14) (see e.g. [11]). □

Remark 1 *As can be expected, the convergence rate of x_k toward x_* is quadratic, if (14) is replaced by*

$$(W_k - H_k)d_k = O(\|d_k\|^2)$$

(see e.g. [3])

5 Fast Local Convergence of SQP Methods

5.1 A Line Search Filter SQP Method

In the companion paper [14] we propose a filter line search SQP method for solving NLPs with bound constraints, for simplicity stated in the form

$$\min_{x \in \mathbb{R}^n} \quad f(x) \tag{47a}$$
$$\text{subject to} \quad c(x) = 0 \tag{47b}$$
$$x \geq 0. \tag{47c}$$

The filter line search algorithm is essentially identical to the one for solving the equality constrained problem (1), where the search direction d_k is now computed as a solution of the QP

$$\min_{d \in \mathbb{R}^n} \quad g_k^T d + \frac{1}{2} d^T H_k d \tag{48a}$$
$$\text{subject to} \quad A_k^T d + c_k = 0 \tag{48b}$$
$$x_k + d \geq 0. \tag{48c}$$

The QP Hessian H_k is assumed to be positive definite in the null space of the constraints active at x_k and $x_k + d_k$. Since the initial iterate as well as all iterates returned from the feasibility restoration phase are assumed to satisfy the bound constraints, we have from (48c) and (5) that $x_k \geq 0$ for all k. Therefore, the infeasibility is still measured as $\theta(x) := \|c(x)\|$. For details, see [14].

In order to achieve fast local convergence for this active set approach, we can again use second order correction steps. One possibility for computing a second order correction step in an SQP framework is proposed in [5], where the composite step $\tilde{d}_k = d_k + d_k^{\text{soc}}$ is obtained as a solution of

$$\min_{\tilde{d}\in\mathbb{R}^n} \quad g_k^T\tilde{d} + \frac{1}{2}\tilde{d}^T H_k\tilde{d} \tag{49a}$$

$$\text{subject to} \quad A_k^T\tilde{d} + c_k + c(x_k + d_k) = 0 \tag{49b}$$

$$x_k + \tilde{d} \geq 0. \tag{49c}$$

This corresponds to the choice (SOC-2) in Section 3.

Let us now assume that the iterates x_k generated by the SQP filter line search algorithm converge to a local solution x_* of (47) satisfying the following second order sufficient conditions.

- There exist multipliers $\lambda_* \in \mathbb{R}^m$ and $v_* \in \mathbb{R}^n$ with $v_* \geq 0$, so that the KKT conditions

$$\begin{array}{rcl} g(x_*) + A(x_*)\lambda_* - v_* & = & 0 \\ c(x_*) & = & 0 \\ v_* \;\geq\; 0, \qquad\qquad x_* & \geq & 0 \\ v_*^T x_* & = & 0 \end{array}$$

  hold;

- the gradients of the constraints active at x_* are linearly independent;

- the Hessian of the Lagrangian, $W_* = \nabla^2 f(x_*) + \sum_{j=1,\ldots,m} \lambda_*^{(j)}\nabla^2 c^{(j)}(x_*)$, is positive definite in the null space of the active constraints;

- strict complementarity holds, i.e. $v_*^{(i)} + x_*^{(i)} > 0$ for all $i = 1, \ldots, n$.

If we assume that the QP Hessians H_k are uniformly positive definite in the null space of the constraints active at x_k and $x_k + d_k$ (see Assumption (G3*) in [14]), then the bound constraints active at x_* are identical to the bound constraints active at the solution of (48) and (49) if x_k is sufficiently close to x_*. Therefore, for large k, the computation of d_k and d_k^{soc} from the QPs (48) and (49) can be interpreted as the steps obtained from Algorithm I applied to an equality constrained NLP, where the equality constraints consist of the equality constraints (47b) and constraints $x^{(i)} = 0$ for $i \in \{j : x_*^{(j)} = 0\}$. As a consequence, the analysis in the previous section can be applied.

5.2 A Trust Region Filter SQP Method

In [6], Fletcher et al. propose a trust region filter SQP algorithm and analyze its global convergence behavior. The switching rule used there does not imply the relationship (26), and therefore the proof of Lemma 4 in our local convergence analysis does not hold for that method. However, it is easy to see that the global convergence analysis in [6] is still valid (in particular Lemma 3.7 and

Lemma 3.10 in [6]), if the switching rule Eq. (2.19) in [6] is modified in analogy to (7) and replaced by

$$m_k(x_k) - m_k(x_k + s_k) \geq 0 \qquad \text{and} \qquad [m_k(x_k) - m_k(x_k + s_k)]^{s_f} \Delta_k^{1-s_f} \geq \kappa_\theta \theta_k^\psi,$$

where m_k is a quadratic model of the objective function, s_k is the trial step, Δ_k is the current trust region radius, $\kappa_\theta, \psi > 0$ constants from [6] satisfying certain relationships, and the new constant $s_f > 0$ satisfies $s_f > 2\psi$. Then the local convergence analysis in Section 4 is still valid (also for the quadratic model formulation), assuming that sufficiently close to a strict local solution the trust region is inactive, the trust region radius Δ_k is uniformly bounded away from zero, the (approximate) SQP steps s_k are computed sufficiently exactly, and a second order correction as discussed in Section 3 is performed.

6 Conclusions

We have shown that second order correction steps are able to overcome the Maratos effect within filter methods and that fast local convergence can be obtained. Important for the success of our analysis is a particular switching rule (7), which differs from previous filter methods, such as the one proposed by Fletcher et al. [6].

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References

[1] P. T. Boggs, J. W. Tolle, and P. Wang. On the local convergence of quasi-Newton methods for constrained optimization. *SIAM Journal on Control and Optimization*, 20:161–171, 1982.

[2] R. M. Chamberlain, C. Lemarechal, H. C. Pedersen, and M. J. D. Powell. The watchdog technique for forcing convergence in algorithms for constrained optimization. *Mathematical Programming Study*, 16:1–17, 1982.

[3] A. R. Conn, N. I. M. Gould, and Ph. L. Toint. *Trust-Region Methods*. SIAM, Philadelphia, PA, USA, 2000.

[4] J. E. Dennis and J. J. Moré. Quasi-Newton methods, motivation and theory. *SIAM Review*, 19(1):46–89, 1977.

[5] R. Fletcher. *Practical Methods of Optimization*. John Wiley and Sons, New York, USA, second edition, 1987.

[6] R. Fletcher, N. I. M. Gould, S. Leyffer, Ph. L. Toint, and A. Wächter. Global convergence of a trust-region SQP-filter algorithms for general nonlinear programming. *SIAM Journal on Optimization*, 13(3):635–659, 2002.

[7] R. Fletcher and S. Leyffer. Nonlinear programming without a penalty function. *Mathematical Programming*, 91(2):239–269, 2002.

[8] R. Fletcher, S. Leyffer, and Ph. L. Toint. On the global convergence of a filter-SQP algorithm. *SIAM Journal on Optimization*, 13(1):44–59, 2002.

[9] S.-P. Han. A globally convergent method for nonlinear programming. *Journal of Optimization Theory and Application*, 22:297–309, 1977.

[10] N. Maratos. *Exact Penalty Function Algorithms for Finite Dimensional and Control Optimization Problems*. PhD thesis, University of London, London, UK, 1978.

[11] J. Nocedal and M. Overton. Projected Hessian updating algorithms for nonlinearly constrained optimization. *SIAM Journal on Numerical Analysis*, 22:821–850, 1985.

[12] J. Nocedal and S. Wright. *Numerical Optimization*. Springer, New York, NY, USA, 1999.

[13] S. Ulbrich. On the superlinear local convergence of a filter-SQP method. *Mathematical Programming*, 100(1):217–245, 2004.

[14] A. Wächter and L. T. Biegler. Line search filter methods for nonlinear programming: Motivation and global convergence. Technical report, Department of Chemical Engineering, Carnegie Mellon University, 2003.