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A Counterexample to a Voronoi Region Conjecture

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***Abstract*—Given a 2D-symmetric lattice Λ, it was conjectured by Forney that the projection of the Voronoi region $\mathcal{R}(\Lambda)$ onto two coordinates equals the Voronoi region of the constituent 2D-sublattice Λ_2'. We present a three-dimensional counterexample.**

***Index Terms*—Voronoi region, lattices.**

I. INTRODUCTION

Let Λ be a lattice (in R^n). Following [1], we define the *Voronoi region* $\mathcal{R}(\Lambda)$ of the lattice Λ as the set of points (in R^n) that are at least as close to the origin as to any other lattice point. Given a pair of coordinates (i,j), $1 \le i \ne j \le n$, we define the *projection* $P_{(i,j)}(\Lambda)$ as the set of points $(x,y) \in \mathrm{R}^2$ such that there exists a point $\boldsymbol{p} \in \Lambda$ with $p_i = x$ and $p_j = y$. If this set is discrete, it is again a (two-dimensional) lattice. We also define the cross-sections $\Lambda'_{(i,j)}$ of a lattice Λ, as the set of points $\{P_{(i,j)}(\boldsymbol{p}) : \boldsymbol{p} \in \Lambda \text{ and } p_k = 0 \text{ for } k \ne i,j\}$. Clearly, $\Lambda'_{(i,j)}$ is a (two-dimensional) lattice and $\Lambda'_{(i,j)} \subseteq P_{(i,j)}(\Lambda)$.

We call a lattice 2D-*symmetric* if, for any pair (i,j), $1 \le i \ne j \le n$

- $P_{(i,j)}(\Lambda) = \Lambda_2$ for some Λ_2, and
- $\Lambda'_{(i,j)} = \Lambda'_2$ for some Λ'_2.

In the case of a 2D-symmetric lattice we can write the projection of $\mathcal{R}(\Lambda)$ onto the coordinate pair (i,j), $P_{(i,j)}(\mathcal{R}(\Lambda))$, as $P_2(\mathcal{R}(\Lambda))$, and $\Lambda'_{(i,j)}$ as Λ'_2. It was shown in [1] that for a 2D-symmetric lattice we have

$$P_2(\mathcal{R}(\Lambda)) \subseteq \mathcal{R}(\Lambda'_2) \tag{1}$$

and it was conjectured that (1) holds with equality. We will now show that, in general, we have strict inclusion.

II. THE COUNTEREXAMPLE

Let $\Lambda \subset \mathrm{Z}^3$ be the lattice generated by the rows of

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & 2 & 1 \end{bmatrix}.$$

For any pair (i,j), $1 \le i,j \le 3$, $P_{(i,j)}(\Lambda) = \mathrm{Z}^2$, and further the cross-section of Λ with any two given coordinates yields the lattice Λ'_2 given by the basis

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}. \tag{2}$$

Therefore, Λ is 2D-symmetric. We next determine the face-defining lattice points which determine the Voronoi region $\mathcal{R}(\Lambda)$ [2]. For a fixed $m \in \mathrm{Z}$, lattice points of the form

$$l(1,-1,0) + k(0,1,-1) + m(2,2,1), \quad l,k \in \mathrm{Z}$$

lie in the plane $\Pi(m)$ given by $x + y + z - 5m = 0$. Note that $\Pi(m), m \in \mathrm{Z}$, is perpendicular to the unit vector $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ and that the distance between two consecutive planes $\Pi(m)$ and $\Pi(m+1)$ is $5/\sqrt{3}$.

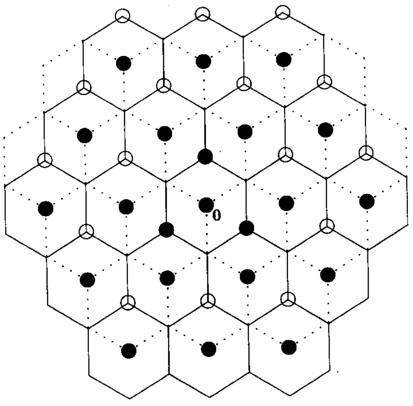

Fig. 1. Cross-sections of $\mathrm{H}(\boldsymbol{q}), \boldsymbol{q} \in \Pi(1) \cap \Lambda$ (dotted lines) and $\mathrm{H}(\boldsymbol{p})$, $\boldsymbol{p} \in \Pi(0) \cap \Lambda$ (solid lines) in $\Pi(0)$. Black dots denote elements of $\Pi(0) \cap \Lambda$, and all other dots denote the projection of elements of $\Pi(1) \cap \Lambda$ in $\Pi(0)$. The projections of $(2,2,1)$ and its cycles are shaded.

We proceed in our search for the face-defining lattice points one plane at a time. First note that $\Pi(0) \cap \Lambda$ is the 2D hexagonal lattice [2, p. 5] with basis

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

and six nearest neighbors at distance $\sqrt{2}$ from the origin. These are the points $\pm(1,-1,0)$ and their cycles. Since $\sqrt{2} < 5/\sqrt{3}$ these are also the nearest neighbors of the origin for the lattice Λ and, hence, must be face-defining lattice points. The six planes defined by these points give rise to a hexagonal cylindrical region, which we shall denote by $\mathrm{H}(\mathbf{0})$. Clearly, $\mathcal{R}(\Lambda) \subseteq \mathrm{H}(\mathbf{0})$.

Analogously we can construct a similar hexagonal cylindrical region $\mathrm{H}(\boldsymbol{p})$ for each point $\boldsymbol{p} \in \Lambda$. Note that if $\boldsymbol{p} \in \Lambda$ is a face-defining lattice point then its Voronoi region $\mathcal{R}(\Lambda) + \boldsymbol{p}$ must share a face with $\mathcal{R}(\Lambda)$ [2, p. 33]. We consider the plane $\Pi(1)$. Fig. 1 shows the cross-sections of $\mathrm{H}(\boldsymbol{q}), \boldsymbol{q} \in \Pi(1) \cap \Lambda$ (dotted lines) and $\mathrm{H}(\boldsymbol{p}), \boldsymbol{p} \in \Pi(0) \cap \Lambda$ (solid lines) in $\Pi(0)$. From this figure it is clear that the only lattice points $\boldsymbol{q} \in \Pi(1) \cap \Lambda$ such that the intersection of $\mathrm{H}(\boldsymbol{q})$ and $\mathrm{H}(\mathbf{0})$ is more than an edge are the points given by $(2,2,1)$ and its cycles (projection of these points is shown by shaded dots). An equivalent analysis shows that the (possibly) face-defining lattice points in the plane $\Pi(-1)$ are given by $(-2,-2,-1)$ and its cycles.

The region defined by the 12 lattice points $\pm(1,-1,0)$, $\pm(2,2,1)$ and their cycles is a convex polytope with 12 faces and 14 vertices (extreme points) at $\pm(1.1, 1.1, 0.1)$, $\pm(0.9, 0.9, 0.9)$, $\pm(1.5, 0.5, 0.5)$, and their cycles. This polytope is contained in a sphere of radius $\sqrt{2.75} = \sqrt{1.5^2 + 0.5^2 + 0.5^2}$. Since any lattice point in a plane $\Pi(m)$, $|m| > 1$, is at least at a distance $2 \cdot 5/\sqrt{3}$ from the origin and $5/\sqrt{3} > \sqrt{2.75}$, we conclude that no other point of Λ can be a face-defining lattice point and, hence, the above region is indeed the Voronoi region $\mathcal{R}(\Lambda)$. The eight shallow holes $\pm(1.1, 1.1, 0.1)$, $\pm(0.9, 0.9, 0.9)$, and their cycles are at squared distance 2.43 and the six deep holes $\pm(1.5, 0.5, 0.5)$ and their cycles are at squared distance 2.75.

Fig. 2 depicts $\mathcal{R}(\Lambda)$ which has the shape of a pencil sharpened at both ends. Also shown is $P_{(x,y)}(\mathcal{R}(\Lambda))$, i.e., the projection of $\mathcal{R}(\Lambda)$ onto the xy-plane. Since $\mathcal{R}(\Lambda)$ is a convex polytope, $P_{(x,y)}(\mathcal{R}(\Lambda))$ can be determined by projecting the extreme points (vertices) of $\mathcal{R}(\Lambda)$ onto the xy-plane and calculating the convex hull. This projection is indicated in Fig. 2 by means of dashed lines.

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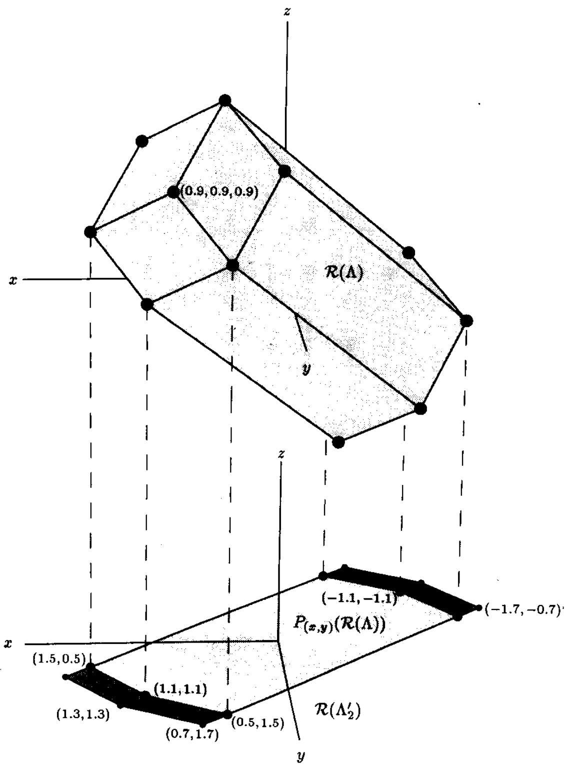

Fig. 2. $\mathcal{R}(\Lambda)$, $P_{(x,y)}(\mathcal{R}(\Lambda))$, and $\mathcal{R}(\Lambda_2')$. The projection onto the xy-plane is indicated by means of dashed lines. The dark gray area denotes points that are contained in $\mathcal{R}(\Lambda_2')$ but not in $P_{(x,y)}(\mathcal{R}(\Lambda))$.

Thus $P_{(x,y)}(\mathcal{R}(\Lambda))$ is the convex polytope with vertices $\pm(1.1, 1.1)$, $\pm(1.5, 0.5)$, and $\pm(0.5, 1.5)$. Finally, Fig. 2 also shows $\mathcal{R}(\Lambda_2')$, which from (2) can be determined to be the convex polytope with vertices $\pm(1.3, 1.3)$, $\pm(1.7, 0.7)$, $\pm(0.7, 1.7)$. Clearly, $\mathcal{R}(\Lambda_2')$ is strictly larger than $P_{(x,y)}(\mathcal{R}(\Lambda))$, which establishes the counterexample.

As a consequence, we mention that the shaping constellation expansion ratio $\mathrm{CER}_s(\Lambda)$, defined by

$$\mathrm{CER}_s(\Lambda) = |P_2(\mathcal{R}(\Lambda))|/|\mathcal{R}(\Lambda)|^{2/n}$$

is, in general, just upper-bounded by and does not necessarily equal $|\mathcal{R}(\Lambda_2)|/|\mathcal{R}(\Lambda)|^{2/n}$.

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REFERENCES

[1] G. D. Forney Jr., "Multidimensional constellations—Part II: Voronoi constellations," *IEEE J. Selected Areas Commun.*, vol. 7, pp. 941–958, Aug. 1989.

[2] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*. New York: Springer-Verlag, 1988.

Invertibility Conditions of a Class of Moment Matrices and Applications

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***Abstract*—We consider a multivariable polynomial with random variables as arguments. All the monoids of this polynomial are arranged to define a random vector for which we study the correlation matrix. We establish necessary and sufficient invertibility conditions of this matrix, and we investigate some applications.**

***Index Terms*— Moment matrices, invertibility conditions, polynomial estimation.**

I. INTRODUCTION

During the last few years, higher order sample moments have been applied to various fields such as blind deconvolution of non-Gaussian linear processes [7], [11], and nonlinear estimation [2], [12]. In fact, higher order sample moments have been studied for many years [1], [3], [13]. For instance, mean square (m.s.) estimation of a random process using second-order continuous-time Volterra filters was studied in [6]. In the same way, polynomial regression was treated in [5] and "*The Problem of Moments*" was investigated in [14].

Volterra filtering of order p based on a finite number of random variables (r.v.s) consists in solving the equation $c = Rh$ where c is fixed, h is the parameter vector, and R is the covariance matrix of a vector whose elements are all possible products of p or fewer r.v.s. Then the elements of R are moments up to order $2p$. Obviously, R is nonnegative definite (n.n.d.), but it is not necessarily invertible. In [12] for instance, R is not invertible. Note that a discrete Volterra filter of order p is a particular multivariable polynomial whose partial degree with respect to each argument is equal to p.

In this correspondence, we consider a multivariable polynomial with arbitrary degrees and with r.v.s as arguments. All the monoids are arranged in a vector $\mathcal{X}$ and we deal with the correlation matrix $\mathcal{K}$ of $\mathcal{X}$. When the degree of the polynomials is equal to one, $\mathcal{K}$ is not invertible if and only if (iff) the r.v.s are linearly dependent. It is interesting to study the same question for arbitrary degrees. The case of a monovariable polynomial was studied in a different context [14]. Moment matrices are topical in statistics [8] and have applications, for instance, in mixture problems [9]. In this work, we establish necessary and sufficient invertibility conditions of $\mathcal{K}$. Then we investigate some applications of these results, in particular in the context of the polynomial estimation of a r.v. or of a deterministic parameter.

II. ASSUMPTIONS AND NOTATIONS

Let $\{x_i\}_{1 \leq i \leq n}$ be real r.v.s and let $\mathcal{P}(\mathcal{X})$ be a real polynomial of $\{x_i\}_{1 \leq i \leq n}$. The partial degree of $\mathcal{P}(\mathcal{X})$ with respect to any x_i is denoted p_i and its total degree is p. Then we have

$$p_i \leq p \leq \sum_{i=1}^{n} p_i$$

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