The Turbo Decoding Algorithm and Its Phase Trajectories

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Abstract — **We analyze phase trajectories of the turbo decoding algorithm as a function of the signal-to-noise ratio (SNR). We prove the existence of fixed points not only at asymptotically high SNRs but also at asymptotically low SNRs. Fixed points at practical SNRs are empirically divided into two classes: indecisive fixed points which usually lead to numerous erroneous decisions and unequivocal fixed points which usually correspond to correct decisions. The waterfall region in the performance curve of turbo decoding is characterized as the region of transition from convergence to indecisive fixed points to convergence to unequivocal fixed points.**

I. Introduction

We consider classical turbo codes, transmitted over an additive white Gaussian noise channel using binary phase-shift-keying modulation. The corresponding turbo decoding algorithm can be viewed as a discrete dynamical system [3]. This dynamical system iteratively updates two probability densities on information bits — commonly known as the *extrinsic information* — provided by the two constituent decoders of the turbo decoding algorithm.

As a dynamical system, the turbo decoding algorithm can have a variety of phase trajectories. A phase trajectory may converge to a fixed point, reach a well-defined invariant set, or simply wander in the high-dimensional space of extrinsic information. At present, preciously little is known about the characteristics of these phase trajectories. For example, in many cases, the turbo decoding algorithm does not converge after a fixed number (say 18) of iterations. Is it possible that in the majority of such cases the decoding algorithm actually converges, albeit only after a large number of iterations? Or is the opposite true: in the majority of such cases, the decoding will never converge. It has been observed that the turbo decoding algorithm always converges at high SNRs. What happens at (asymptotically) low SNRs: Does the algorithm converge or does it wander ad infinitum? These are some of the basic questions answered in this work.

II. Fixed Points at Asymptotic SNRs

Using a set of sufficient conditions provided by Richardson [3], we show [1] that at asymptotically low SNRs, with high probability, the turbo decoding algorithm has a unique fixed point. The extrinsic information that corresponds to this fixed point is close to the uniform distribution on information bits. That is, the fixed point votes almost equally in favor of the two possible values for each transmitted information bit.

On the other hand, we show that at asymptotically high SNRs, with high probability, the turbo decoding algorithm has fixed points that correspond to the transmitted codeword. Moreover, starting from unbiased initialization, the turbo decoding algorithm will converge to one of these fixed points. The derivation of this result indicates that extrinsic information corresponding to such fixed points is concentrated on the transmitted information bits.

III. Fixed Points at Practical SNRs

The existence of certain fixed points at *asymptotic* SNRs raises interesting questions. Does the turbo decoding algorithm, starting from unbiased initialization, converge to these fixed points? If so, what are the threshold values of SNR beyond which the turbo decoding algorithm converges?

To answer these questions, we performed extensive simulations. Empirically, we found that the turbo decoding algorithm converges to two types of fixed points: *indecisive* fixed points and *unequivocal* fixed points. The algorithm converges to indecisive fixed points for SNRs that are below the waterfall region, and to unequivocal fixed points for SNRs above the waterfall region. The empirically observed characteristics of indecisive and unequivocal fixed points match closely the characteristics predicted by our analysis for asymptotically low and asymptotically high SNRs, respectively.

For SNRs in the waterfall region, the decoding algorithm may or may not converge, and in some cases, the phase trajectory may become quasi-period or periodic.

IV. Continuation of Fixed Points

For sufficiently long turbo codes, we can treat the turbo decoding algorithm as a single-parameter dynamical system, parameterized (approximately) by the SNR. This allows us to trace the movement of fixed points (more precisely, obtain the *equilibrium curves* of fixed points) as the SNR is changed.

The equilibrium curves, parameterized (approximately) by the SNR, reveal that unequivocal fixed points barely move as the SNR is changed from very high to very low values. However, starting from the very low values, indecisive fixed points move substantially as the SNR is increased while becoming less and less stable. Ultimately, for SNRs in the waterfall region, indecisive fixed points bifurcate and disappear. All three types of bifurcation, studied in classical bifurcation theory [2], occur in turbo decoding. This explains the quasi-periodic and periodic behavior of the phase trajectories in the waterfall region.

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References

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