Generalized Minimum-Distance Decoding in Euclidean-Space: Performance Analysis

Dakshi Agrawal and Alexander Vardy

Coordinated Science Laboratory, University of Illinois, 1308 W. Main Street, Urbana, IL 61801, USA
{dakshi,vardy}@golay.csl.uiuc.edu

Abstract — **Detailed geometric analysis of decoding regions for GMD decoding is presented. We show that GMD decoding regions are non-convex in essentially all cases of interest. We also prove that these regions are always bounded by hyperplane segments. For d-erasure GMD decoding, we establish the presence of a large number of bounding hyperplanes at distance $\le d+1$. These results invalidate the estimates of performance derived from the union bound in the case of (multistage) GMD decoding. Alternative probabilistic estimates of, and upper bounds upon, the performance of GMD decoding are developed. Simulation results, for both low-dimensional and high-dimensional sphere packings, are in remarkably close agreement with our probabilistic approximations.**

I. Introduction

Multistage GMD decoding in Euclidean space was recently proved [1] to be a bounded-distance decoding algorithm up to the true packing radius ρ of the corresponding code or lattice. Furthermore, it was shown that a variant of the GMD decoding algorithm, called d-erasure GMD decoding, has the same effective error-coefficient $\mathcal{N}_{\text{eff}}$ as maximum-likelihood (ML) decoding. Although ρ and $\mathcal{N}_{\text{eff}}$ indeed determine the performance of a decoding algorithm as SNR$\to\infty$, at SNRs of practical interest the significance of $\mathcal{N}_{\text{eff}}$ and ρ derives from the union-bound esitmate. This estimate is based on the premise that the decoding regions are convex and bounded by hyperplanes, which are either at a squared distance d or substantially larger than d from a codeword.

II. Geometric Analysis

First, we prove that, in essentially all cases of interest, GMD decoding regions are bounded by the hyperplane segments, alleviating the concern that in general this may not be the case [1]. We further show that, with the exception of the trivial case $d \le 2$, single-stage GMD decoding regions are *always* non-convex. For the d-erasure modifications of GMD decoding [1], we establish the existence of

$$\binom{n}{d+1}\tau_0(\Lambda)^{d+1}$$

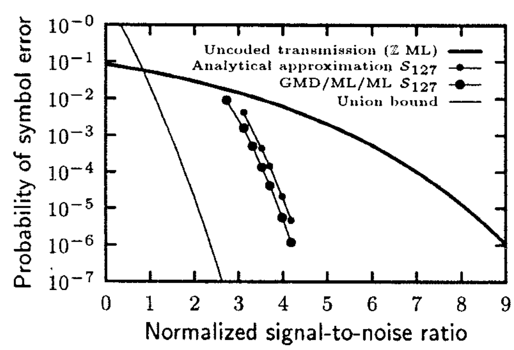
boundary points at squared distance (d+1), where $\tau_0(\Lambda)$ is the kissing number of the ubderlying lattice Λ. Non-convexity of GMD decoding regions and the presence of a large number of boundary points at squared distance only slightly greater than $\rho^2 = d$ indicate that performance analysis based on the union-bound estimate is inapplicable in the case of GMD decoding. This conclusion is confirmed by the simulation results below.

III. Probabilistic Analysis

As an alternative approach, we develop *probabilistic* estimates of the performance of GMD decoding algorithms in Euclidean space. First, we prove that the probability of error at the selection stage of a GMD decoder is usually negligible relative to the overall error probability. Thus the performance of GMD decoding is dominated by the performance of the algebraic decoding stage. We therefore develop probabilistic approximations of the performance of the algebraic decoding stage. In fact, these approximations also constitute precise upper bounds for GMD decoding of block codes.

IV. Simulation Results

The actual performance of multistage GMD decoding algorithms is determined through comprehensive computer simulations. Simulation results show that the performance of multistage GMD decoding is reasonably close (within about 0.3 dB) to ML decoding for the Barness-Wall lattice BW_{16}.



We have also performed comprehensive simulations for the high-dimensional sphere-packing

$$\mathcal{S}_{127} = (127{,}71{,}19)\otimes[\mathbb{Z}/2\mathbb{Z}] + (127{,}126{,}2)\otimes[2\mathbb{Z}/4\mathbb{Z}] + 4\mathbb{Z}^{127}$$

considered in [1]. Note that the only computationally *feasible* way to decode $\mathcal{S}_{127}$ is multistage GMD decoding. The simulation results for $\mathcal{S}_{127}$ demonstrate the advantage of multistage GMD decoding in high dimensions: we obtain a gain of about 5 dB (including the shaping gain), relative to uncoded transmission, as can be seen from the above figure. Finally, we have compared the performance of BW_{16} and $\mathcal{S}_{127}$ obtained through simulations with the probabilistic approximations developed in the paper. In both cases, our approximations are within 0.2 dB of the actual performance.

References

[1] G.D. Forney, Jr. and A. Vardy, "Generalized minimum-distance decoding of Euclidean-space codes and lattices," *IEEE Trans. Inform. Theory*, vol. 42, pp. 1992-2026, 1996.

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