Extreme Fluctuations in Small Worlds with Relaxational Dynamics

H. Guclu, B. Kozma, G. Korniss (Rensselaer)
M.B. Hastings, Z. Toroczkai (Los Alamos)
M.A. Novotny (Mississippi State University)
P.A. Rikvold (Florida State University)

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Synchronization in Parallel Discrete-Event Simulations

Parallelization for asynchronous dynamics

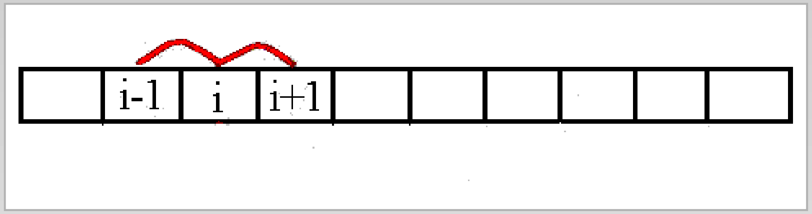
Examples:

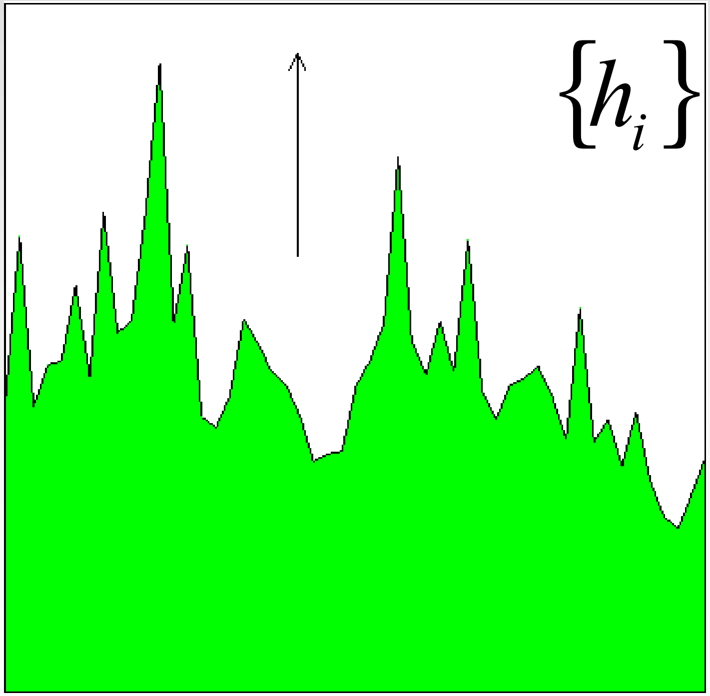
- Cellular communication networks (call arrivals)
- Magnetization dynamics in condensed matter (Ising model with single-spin flip Glauber dynamic)
- Spatial epidemic models, contact process (infections)

Paradoxical task:

- (algorithmically) parallelize (physically) non-parallel dynamics

Conservative synchronization





i (PE index)

- one-site-per PE, $N_{PE}=N=L^d$
- t=0,1,2,… parallel steps
- $h_i(t)$ local simulated time
- local time increments are iid exponential random variables (Poisson asynchrony)
- advance only if $h_i \leq \min\{h_{nn}\}$

(nn: nearest neighbors)

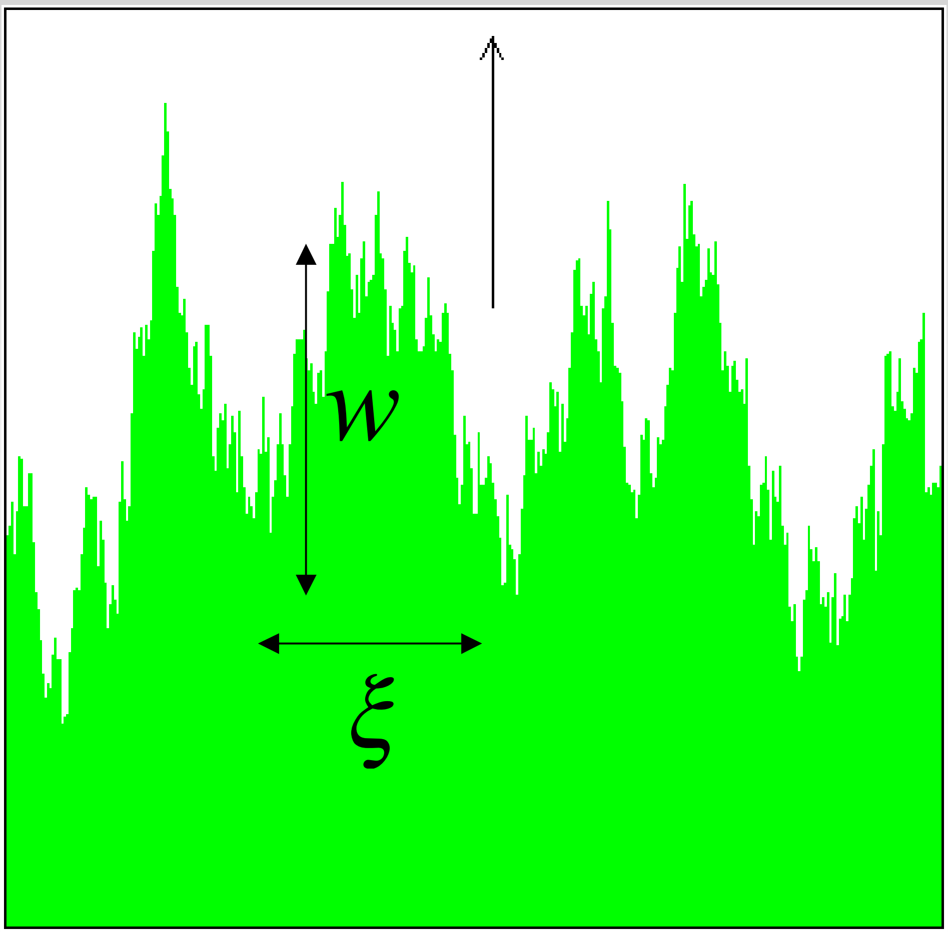
Lubachevsky (1987)

$\{h_i\}$: "virtual time horizon"

"Rough" virtual times

width: $$w^2(t) = \frac{1}{N}\sum_{i=1}^{N}[h_i(t) - \bar{h}(t)]^2$$

$$\bar{h} = \frac{1}{N}\sum_{i=1}^{N} h_i$$



$$N = 10^4$$

"simulation of the simulations"

$$\xi \sim N$$

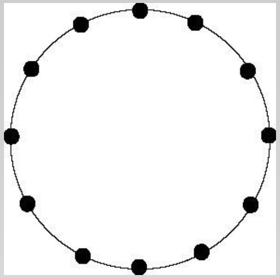
correlation length

$$w \sim N^{\alpha}$$

roughness exponent

Edward-Wilkinson model on a ring

$$\partial_t h_i = (h_{i+1} + h_{i-1} - 2h_i) + \ldots + \eta_i(t)$$



$$\langle \eta_i(t)\eta_j(t')\rangle = 2\delta_{ij}\delta(t-t')$$

Gaussian noise

$$\Gamma^o_{ij} = 2\delta_{i\,j} - \delta_{i\,j-1} - \delta_{i\,j+1}$$

-Laplacian on regular network (ring)

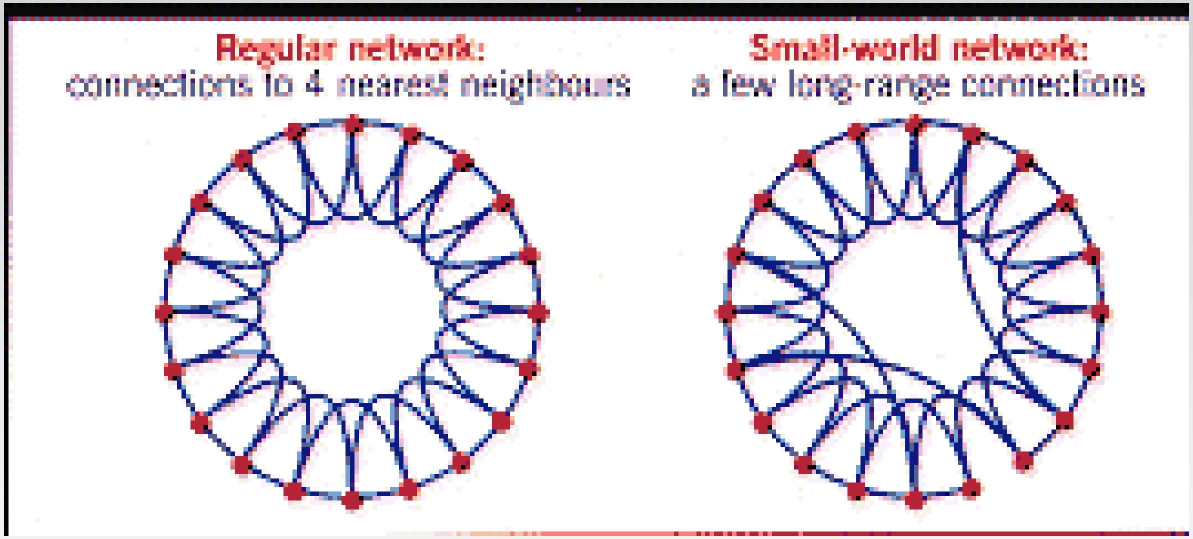
$$\partial_t h_i = -\sum_{j=1}^{N} \Gamma^o_{ij} h_j + \eta_i(t)$$

$$\langle w^2 \rangle_L \sim N^{2\alpha}$$

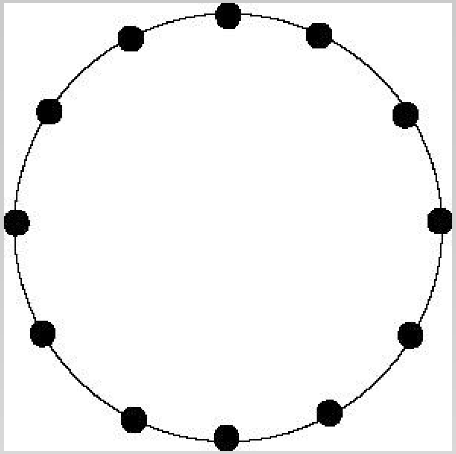
$$\alpha = 1/2$$

roughness exponent

Suppressing roughness in virtual time horizons



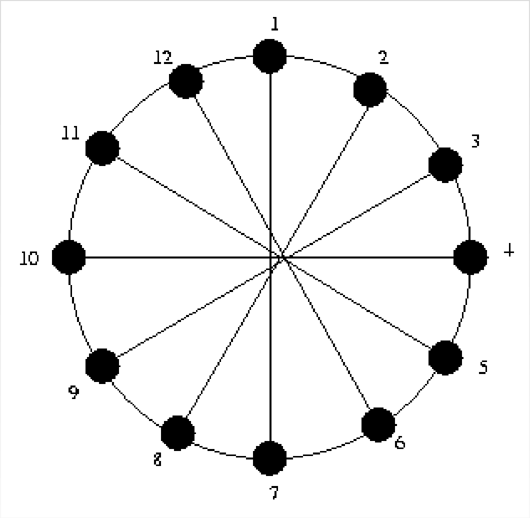
- Watts&Strogatz (1998):
"… enhanced signal-propagation speed, computational power, and synchronizability".



N sites (N=N_{PE})

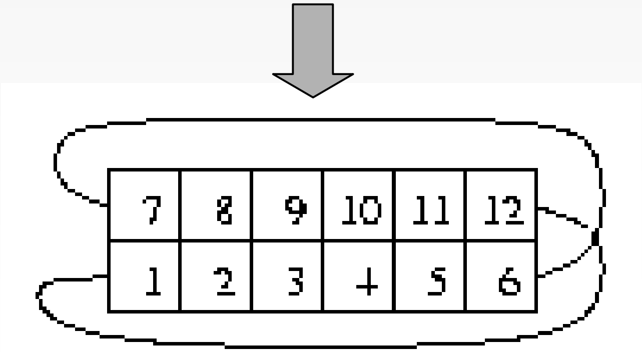
only nn. interactions with p.b.c

$$\langle w^2 \rangle_N = N/12$$



nn. + regular “long-range” interactions (N/2 links)

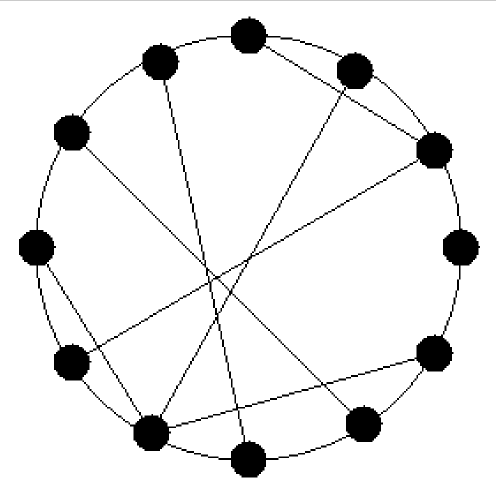
$$\langle w^2 \rangle_N \cong N/24$$



$2\times(N/2)$ with h.b.c.

quasi one-dimensional system

Newman & Watts, Monasson



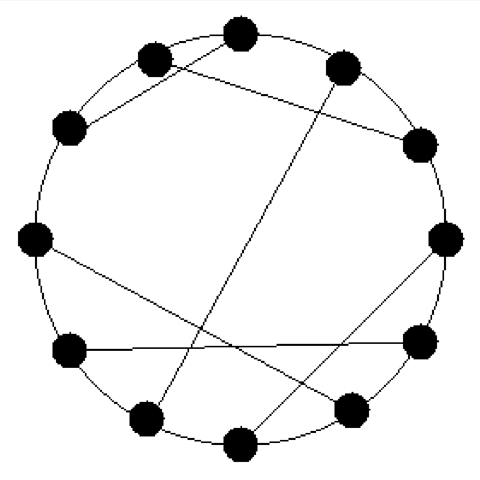
"soft" SW network (ER on top of ring)

$$J_{ij} = \begin{cases} 1 & \text{with probability} \quad p/N \\ 0 & \text{with probability} \quad 1 - p/N \end{cases}$$

$$\left[\sum_l J_{il} \right] = \sum_l [J_{il}] = p$$

average degree
(in addition to n.n.)

average over network realizations



"hard" SW network

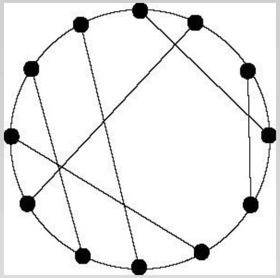
$$J_{ij} = 0, p$$

$N/2$ random links are selected, such that each site has exactly one random neighbor
(in addition to n.n.)

$$\sum_l J_{il} = p$$

no fluctuations
in the individual degree

Edward-Wilkinson (EW) model on small-world (SW) networks



$$\partial_t h_i = (h_{i+1} + h_{i-1} - 2h_i) - \sum_{j=1}^{N} J_{ij}(h_i - h_j) + \eta_i(t)$$

$$\Gamma_{ij}^o = 2\delta_{i\,j} - \delta_{i\,j+1} - \delta_{i\,j-1}$$

-Laplacian on regular network (ring)

$$V_{ij} = \delta_{ij} \sum_l J_{il} - J_{ij}$$

-Laplacian on random part of the network

$$\partial_t h_i = -\sum_{j=1}^{N} \Gamma_{ij} h_j + \eta_i(t)$$

$$\Gamma = \Gamma^o + V$$

Mean-field approximation

$$\partial_t h_i = -\sum_{j=1}^{N} [\Gamma_{ij}] h_j + \eta_i(t) \qquad \Gamma = \Gamma^o + V$$

$$[V_{ij}] = \delta_{ij} \sum_l [J_{il}] - [J_{ij}] = \delta_{ij} p - p/N$$

$$\partial_t h_i = (h_{i+1} + h_{i-1} - 2h_i) - p(h_i - \bar{h}) + \eta_i(t)$$

$$\xi \overset{N\to\infty}{\sim} \frac{1}{\sqrt{p}}$$

$$\langle w^2 \rangle_N \overset{N\to\infty}{\sim} \frac{1}{2\sqrt{p}}$$

Roughness and the spectrum

$$\Delta_i \equiv h_i - \bar{h}$$

height fluctuations, relative to the mean

$$\bar{h} = \frac{1}{N}\sum_{i=1}^{N} h_i$$

for a given realization of the random network:

$$\langle w^2 \rangle_N = \left\langle \frac{1}{N}\sum_{i=1}^{N}(h_i - \bar{h})^2 \right\rangle = \left\langle \frac{1}{N}\sum_{i=1}^{N}\Delta_i^2 \right\rangle = \frac{1}{N}\sum_{l=1}^{N-1}\frac{1}{\lambda_l}$$

$\{\lambda_l\}_{l=0}^{N-1}$: eigenvalues of Γ $(\lambda_0 = 0)$

“disorder” average (average over network realizations)   density of states

$$\left[\langle w^2\rangle_N\right]=\left[\frac{1}{N}\sum_{l=1}^{N-1}\frac{1}{\lambda_l}\right]\xrightarrow{N\to\infty}\int\frac{\rho(\lambda)d\lambda}{\lambda}$$

- $p=0$: $\rho(\lambda)\to 1/\sqrt{\lambda}$ for $\lambda\to 0$, integral and $\langle w^2\rangle$ *diverges*
- sufficiently fast vanishing $\rho(\lambda)$ as $\lambda\to 0$ $\Rightarrow$ *finite* $[\langle w^2\rangle]$

“soft” network: “pseudo-gap” $\Sigma\sim p^2$

$$\rho(\lambda)\sim\left(1/\sqrt{\lambda}\right)e^{-cp/\sqrt{\lambda}}$$

Bray & Rodgers (1988); Monasson (1999)
(in the context of diffusion)

$$\left[\langle w^2\rangle_\infty\right]\sim\frac{1}{p}$$

Kozma, Hastings, G.K. (2003)
(applied to the width)

Perturbative method

$$G_{ij} = \langle \Delta_i \Delta_j \rangle = (\Gamma^{-1})_{ij}$$ two-point correlation function

$$G^o = {\Gamma^o}^{-1} \qquad G = \Gamma^{-1} \qquad \Gamma = \Gamma^o + V$$

$$\left[\langle w^2 \rangle_N\right] = \frac{1}{N}\sum_{i=1}^{N}\left[\langle \Delta_i^2 \rangle\right] = [G_{ii}] = \left[\left(\Gamma^{-1}\right)_{ii}\right]$$

$[G]$ disorder-averaged two-point function

$$[G] = G^o - [G^o V G]$$

$$= G^o - [G^o V G^o] + [G^o V G^o V G^o] - \ldots$$

$$[G]^{-1} = \Gamma^{o} + \Sigma$$

$$\Sigma = [G]^{-1} - G^{o^{-1}}$$

self-energy (effective interaction due to random links)

"soft" network: $\Sigma \sim p^{2} + \ldots$

"hard" network: $\Sigma \sim p - \frac{1}{2} p^{3/2} + \ldots$

$$\partial_t h_i = (h_{i+1} + h_{i-1} - 2h_i) - \Sigma(h_i - \bar{h}) + \eta_i(t)$$

$$\xi \overset{N\to\infty}{\sim} \frac{1}{\sqrt{\Sigma}}$$

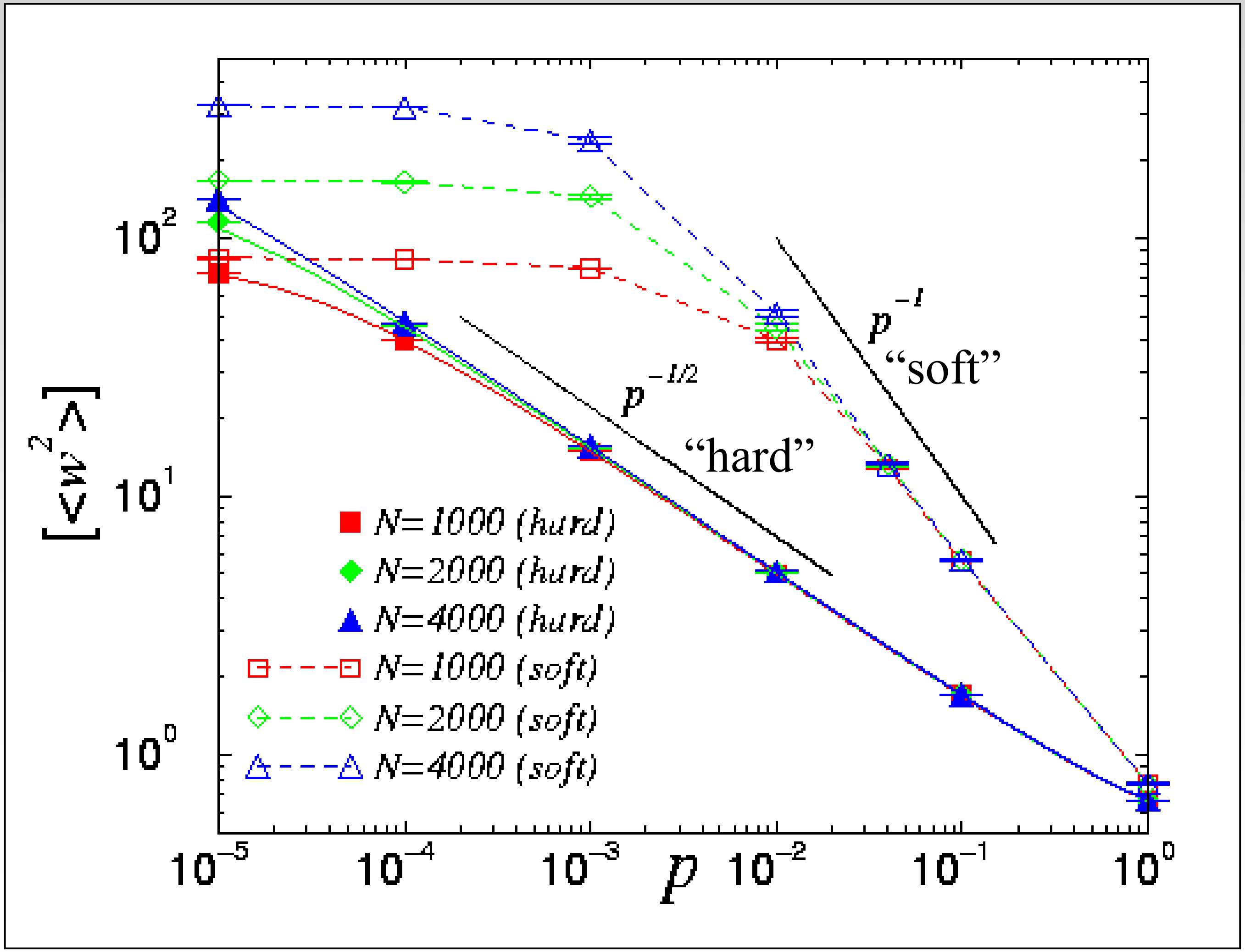
finite correlation length

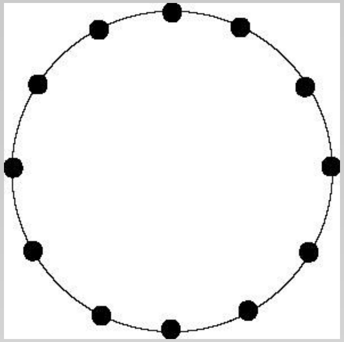
$$\left[\langle w^2 \rangle_N\right] \overset{N\to\infty}{\sim} \frac{1}{2\sqrt{\Sigma}}$$

finite width

for any $p>0$

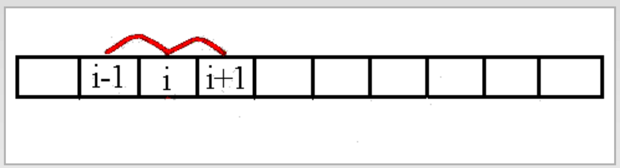
Exact numerical diagonalization results for the EW model on SW networks

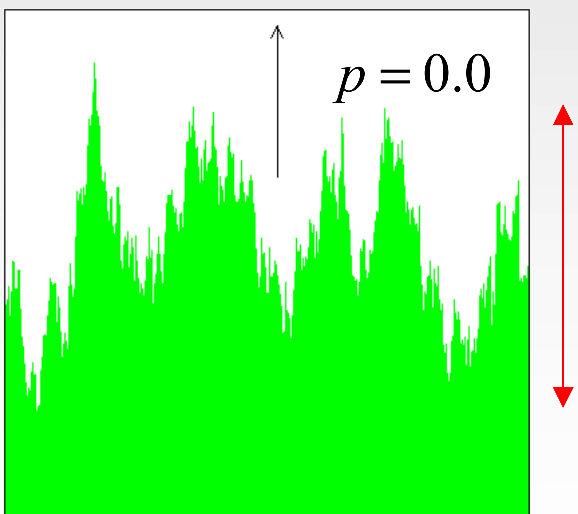




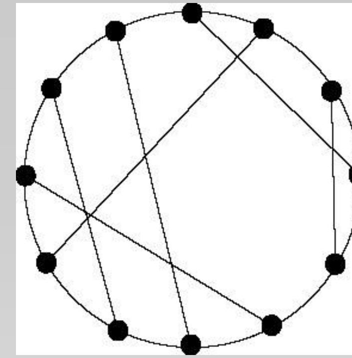
regular lattice (ring) topology

("$p=0$")



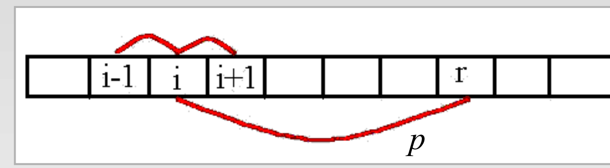


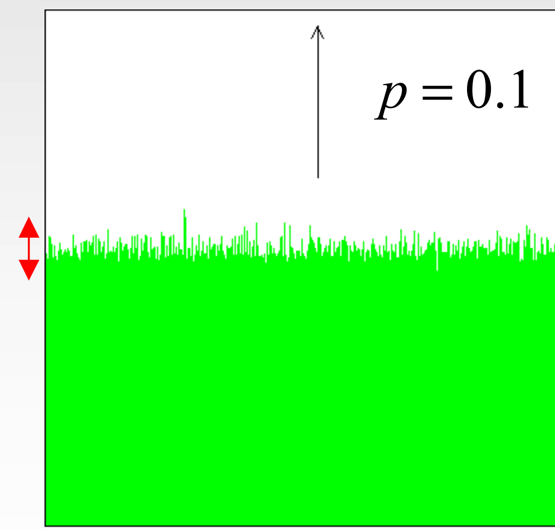
$w \sim N^{\alpha}$



small-world-like connections:

used with probability $p>0$



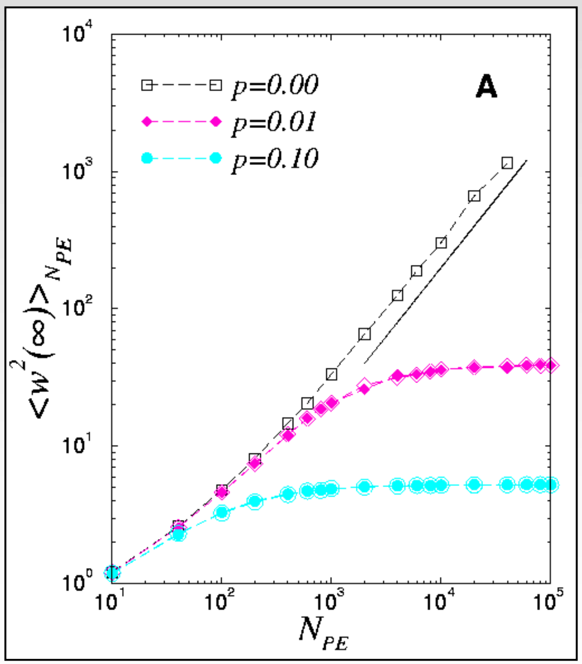


$N = 10^4$

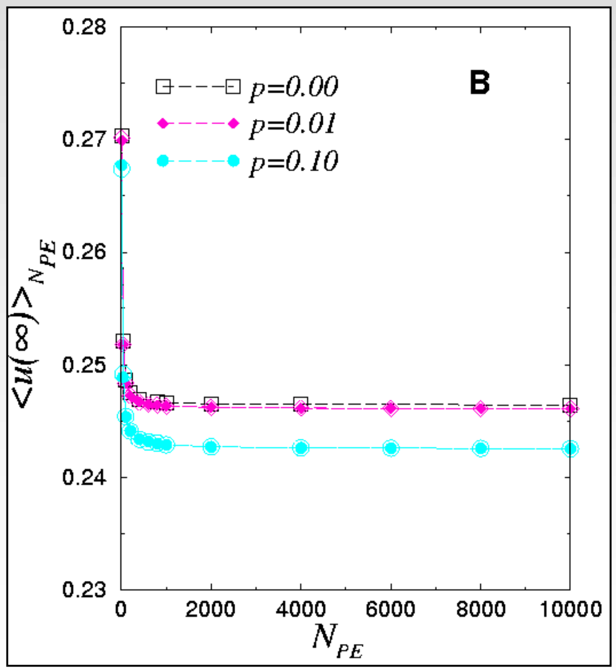
$w \underset{N\to\infty}{\sim} const.$

utilization trade-off/scalable data management

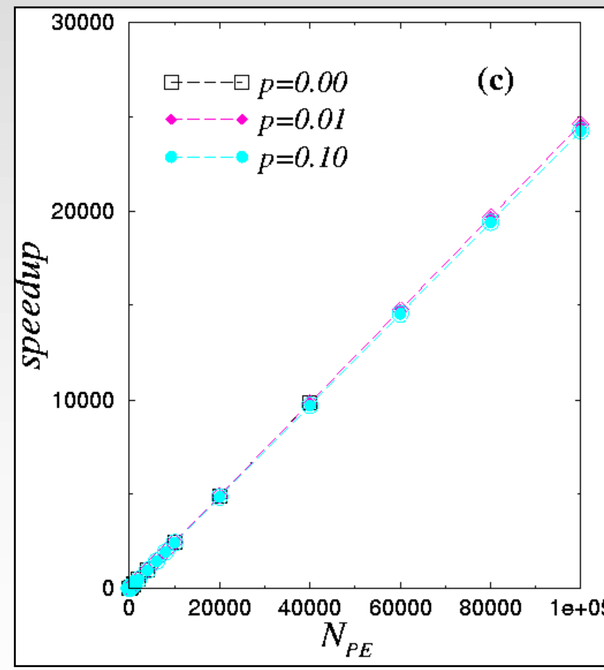
roughness



utilization = fraction of non-idling PEs



speedup = utilization × N_{PE}



G.K. et al. (2003)

Extreme value statistics

N iid variables, $\{\zeta_i\}_{i=1}^{N}$ $P_<(x) = \text{Prob}\{\zeta_\text{i} < x\}$

$\zeta_{\max} = \max\{\zeta_i\}_{i=1}^{N}$ $P_<^{\max}(x) = \text{Prob}\{\zeta_{\max} < x\}$

$$P_<^{\max}(x) = [P_<(x)]^N = [1 - P_>(x)]^N \approx e^{-NP_>(x)}$$

$P_>(x) = e^{-cx^\delta}$ short-tailed individual variables

$$\tilde{x} = (x - a_N) / b_N$$

$$\tilde{P}_{<}^{\max}(\tilde{x}) = e^{-e^{-\tilde{x}}}$$

Fisher-Tippett-Gumbel distr.

$$\langle \tilde{x} \rangle = \gamma \qquad \sigma_{\tilde{x}}^2 = \frac{\pi^2}{6}$$

$$a_N = \left(\ln(N)/c\right)^{1/\delta}$$

$$b_N = (\delta c)^{-1}\left(\ln(N)/c\right)^{1/\delta - 1}$$

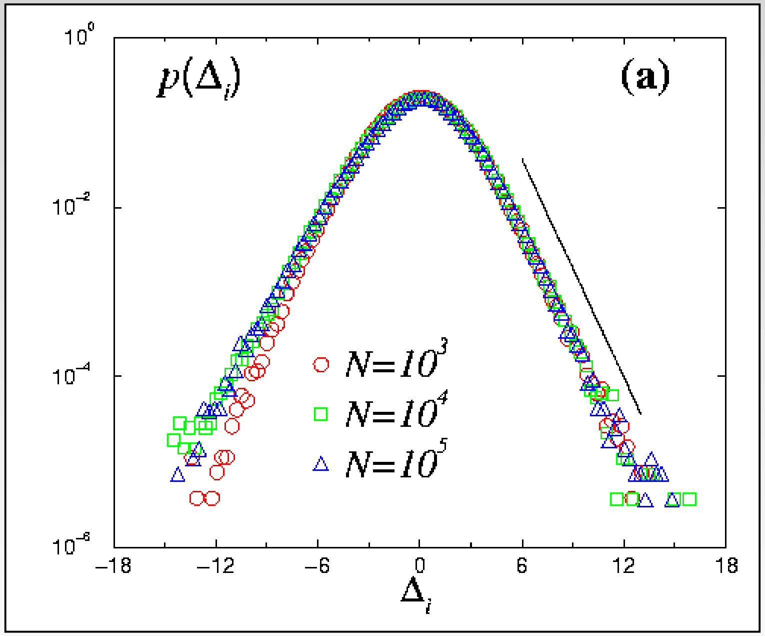
$$\langle \zeta_{\max} \rangle_N = a_N + b_N \gamma \approx a_N = (\ln(N)/c)^{1/\delta}$$

virtual time horizon: $\Delta_i \equiv h_i - \bar{h}$

finite correlation length → quasi-independent blocks

$$N \rightarrow N/\xi$$

Bouchaud & Mézard (1997)
Baldassarri (2000)



$\delta = 1$ (measured)

$$<\Delta_{\max}> \sim w(\ln(N/\xi))^{1/\delta} \overset{N\rightarrow\infty}{\sim} w\ln(N)$$

Extreme height fluctuations in the virtual time horizon

$$\Delta_{\max} = h_{\max} - \bar{h}$$

ring:

$$w \sim N^{\alpha}$$

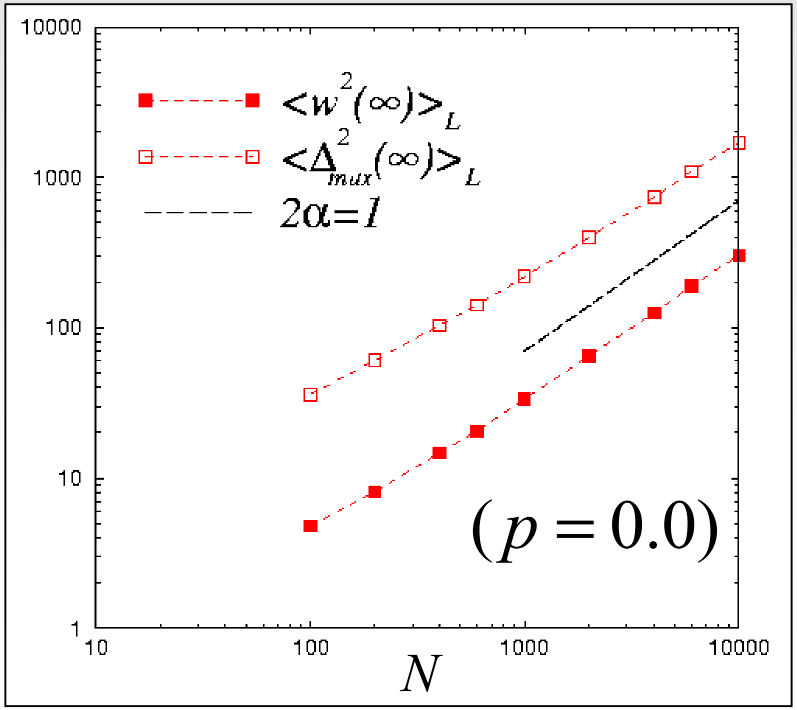
same exponent

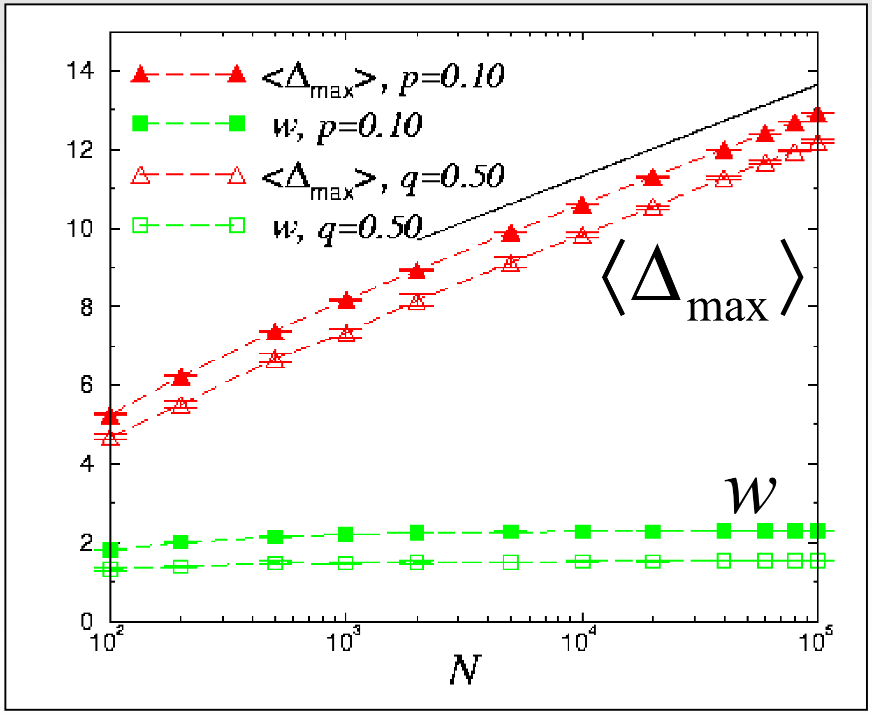
$$\Delta_{\max} \sim N^{\alpha}$$

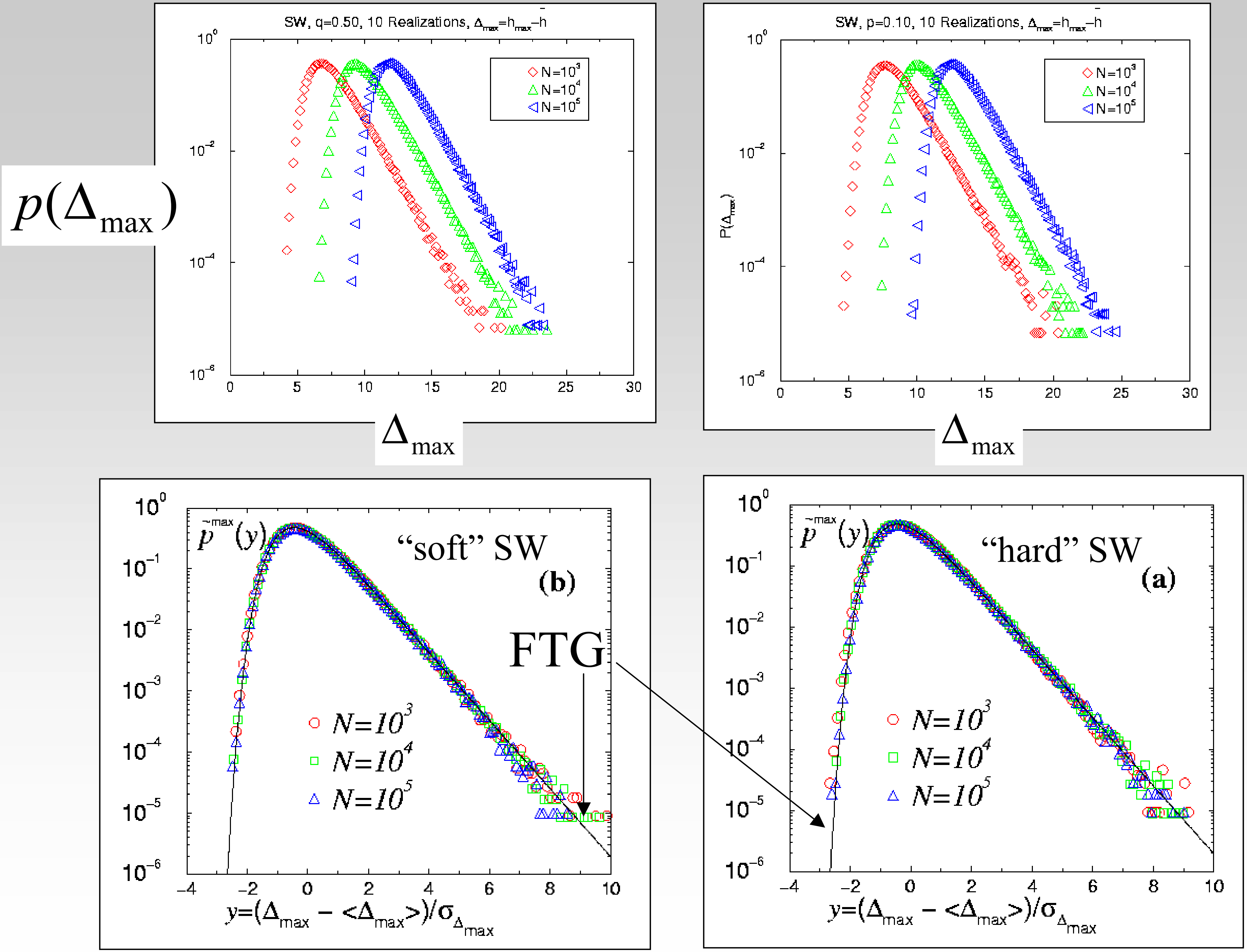
small worlds:

$$w \overset{N\to\infty}{\sim} const.$$

$$\Delta_{\max} \sim \ln(N)$$







SW, q=0.50, 10 Realizations, Δmax=hmax−h
N=10^3
N=10^4
N=10^5
p(Δmax)
Δmax
SW, p=0.10, 10 Realizations, Δmax=hmax−h
N=10^3
N=10^4
N=10^5
P(Δmax)
Δmax
p̃max(y)
"soft" SW
(b)
FTG
N=10^3
N=10^4
N=10^5
y=(Δmax − <Δmax>)/σΔmax
p̃max(y)
"hard" SW
(a)
N=10^3
N=10^4
N=10^5
y=(Δmax − <Δmax>)/σΔmax

Summary and Outlook

- SW links can facilitate synchronization of parallel computing networks with many nodes
- Relevant node-to-node process is relaxation

  → FTG for extreme fluctuations (weak log divergence)
- Outlook: extreme fluctuations of the “load” in other types of networks with other types of noise

www.rpi.edu/~korniss

H. Guclu, and G.K, arXiv:cond-mat/0311575 (2003).

B. Kozma, M.B. Hastings, and G.K, arXiv:cond-mat/0309196 (2003).

G. K., M.A. Novotny, H. Guclu, Z. Toroczkai, and P.A. Rikvold, *Science* **299**, 677 (2003)..

G. K, Z. Toroczkai, M.A. Novotny, and P.A. Rikvold, *Physical Review Letters* **84**, 1351 (2000).