A Very Brief Tutorial on Network Models

Cristopher Moore
Computer Science, University of New Mexico

Model Zero: Erdös-Rényi

- Simplest possible model: uniformly random
- Each pair of vertices independently connected with equal probability p
- n vertices, average degree

$$d = p(n-1) \approx pn$$

- Percolation transition, or "Emergence of a giant component," at $d = 1$

Model Zero: Erdös-Rényi

- Branching process with average ratio d
- If $d < 1$, total expected number of descendants is

$$\frac{1}{1-d}$$

- If $d > 1$, it diverges

Model Zero: Erdös-Rényi

- If $d < 1$, then with high probability:
  - Mostly trees
  - Constant number of unicycles
  - No bicycles
  - No component larger than $\log n$

Model Zero: Erdös-Rényi

- If $d > 1$, then with high probability:
  - A unique "giant component" of size an

    where $1 - a = \mathrm{e}^{-ad}$
  - The rest of the graph looks subcritical,

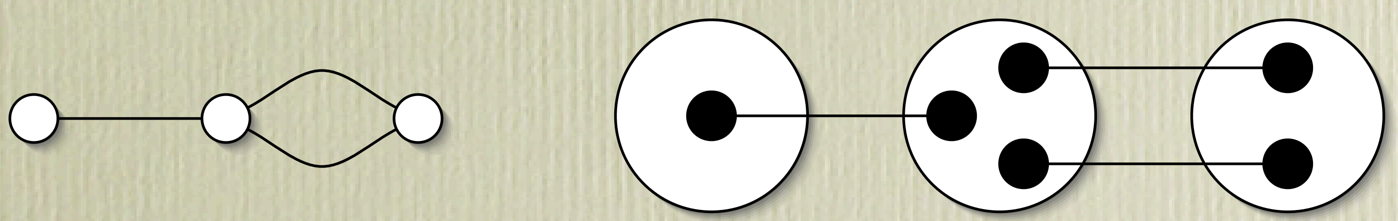
    with $d\mathrm{e}^{-d} = d'\mathrm{e}^{-d'}$

The Degree Distribution P(k)

- Fraction of vertices with k neighbors
- For Erdös-Rényi, Poisson with mean d: $\dfrac{e^{-d} d^k}{k!}$
- Sharply concentrated around its mean
- But most real networks have heavy tails, often with diverging variance

Random graphs with specified degree distributions

- Bender & Canfield, Bollobas
- Percolation studied by Molloy & Reed, Newman
- *Configuration model*: random matching of “copies” or “spokes”



Random graphs with specified degree distributions

- Probability a random edge "hits" a vertex is proportional to its degree

- Average number of outgoing edges from there is

$$\lambda = \frac{\sum_k k(k-1)P(k)}{\sum_k kP(k)}$$

- Percolates when $\lambda > 1$, or equivalently when

$$P^{(2)} - 2P^{(1)} > 0$$

Preferential Attachment

- A mechanism for power laws, with a long history
- Simon: Pareto distributions, “the rich get richer”
- de Solla Price: citation networks
- Albert & Barabási: models of the Web
- In CS theory, Chung and many others

Preferential Attachment

- Many variations... let's take one: at each step, a new vertex is added, with one undirected edge

$$\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{k}{2t} \Rightarrow k = At^{1/2}$$

$$k = 1 \text{ at } t_{\mathrm{birth}}, \text{ so } t_{\mathrm{birth}} = A^{-2}$$

$$P(A) = P(t_{\mathrm{birth}}) \left| \frac{\mathrm{d}t_{\mathrm{birth}}}{\mathrm{d}A} \right| \sim A^{-3}$$

Preferential Attachment

- Drinea, Frieze, Mitzenmacher, Spencer: if we attach with probability proportional to

  $$k^\alpha \text{ for } \alpha > 1$$

  then eventually one person gets all the links.

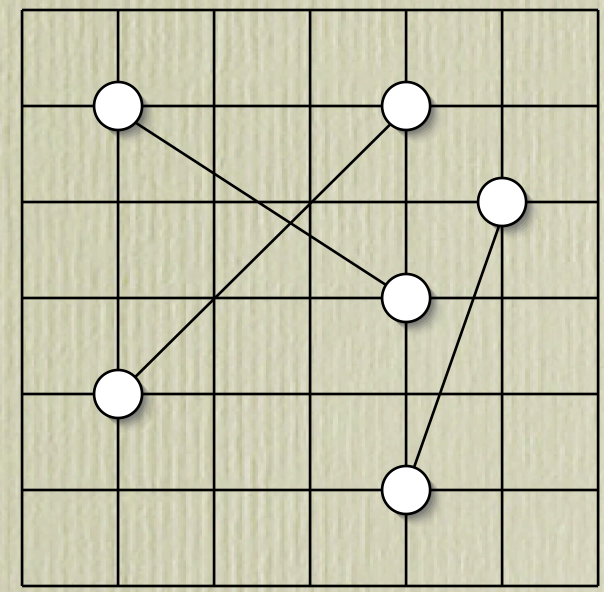
- Spencer: there is a finite time after which *no one else gets anything!* (Very nice argument.)

Getting more realistic...

- Spatial & geometric structure
- Navigability / Searchability
- Community structure and hierarchy

Spatial Structure

- Watts & Strogatz: local + uniform



- Kleinberg: local + power law

Navigability

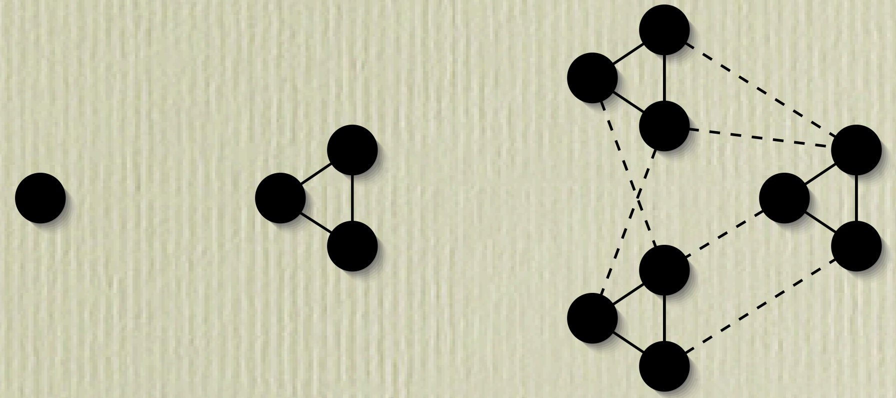
- Small diameter means that short paths exist, but...
- Local information seems to allow us to route efficiently, i.e. we can *find* the short paths
- Kleinberg: power-law distribution of link lengths
- Clauset & Moore: dynamical process that builds this distribution

Community Structure

- Top-down: find vertices of high "betweenness," and break the network apart
- Bottom-up: merge nodes together to maximize "modularity," i.e. ratio between within-cluster edges and between-cluster edges
- Girvan & Newman

Hierarchical structures

- Iterative models like



- Not much known!