Price Formation in Double Auctions*

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Abstract. We develop a model of information processing and strategy choice for participants in a double auction. Sellers in this model form beliefs that an offer will be accepted by some buyer. Similarly, buyers form beliefs that a bid will be accepted. These beliefs are formed on the basis of observed market data, including frequencies of asks, bids, accepted asks, and accepted bids. Then traders choose an action that maximizes their own expected surplus. The trading activity resulting from these beliefs and strategies is sufficient to achieve transaction prices at competitive equilibrium and complete market efficiency after several periods of trading.

1 Introduction

The double auction (DA) is one of the most common exchange institutions, used extensively in stock markets such as the New York Stock Exchange, commodity markets such as the Chicago Mercantile Exchange, and in markets for financial instruments, including options and futures. The prevalence of this institution can be traced to its operational simplicity, efficiency, and to its capacity to respond quickly to changing market conditions. Nevertheless, the DA is a persistent puzzle in economic theory. How is information which is held separately by many market participants – in the form of privately known reservation values and marginal costs – quickly and accurately coordinated through the trading process in order to reach the competitive equilibrium (CE) price and allocation?

In the double auction, any seller may at any time (during a specified trading period) submit an offer that is then observed simultaneously by all buyers and sellers. Similarly, any buyer may submit a bid which is observed by the other

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buyers and by the sellers. When a buyer's bid is acceptable to some seller, that seller may then accept the buyer's bid, and a trade is executed between the buyer whose bid was accepted and the seller who accepted this bid. Similarly, buyers may accept a seller's offer at any time.

Market experiments have established that transaction prices converge quickly to a competitive equilibrium price in the DA for a wide variety of market environments. Experimental investigation of trader behavior and market performance in the DA began with Smith [12]. Smith induced supply and demand conditions by giving buyers a redemption value for each unit of an abstract commodity purchased, and by giving sellers a cost for each unit of this abstract commodity sold. Buyers receive surplus equal to the difference between their redemption value and the purchase price negotiated with a seller, and sellers receive surplus equal to the difference between the purchase price paid by the buyer and their unit cost. Since reservation prices – and therefore supply and demand conditions – are known to the experimenter when this procedure is employed, the procedure makes possible comparison between experimental outcomes and theoretical predictions. The basic result observed in these experiments is that prices do converge quickly to within a few cents of competitive equilibrium prices in markets with stationary supply and demand. Smith and many other economists in the 35 years since his initial studies have also documented features of the path of convergence to equilibrium in a variety of market environments. For surveys and interpretation of these experimental results, see Plott [11] and Smith [13].

Models of trader behavior in the DA have been constructed by several authors, including Easley and Ledyard [3], Friedman [4], Gode and Sunder [7], and Wilson [14]. Although these models have furthered understanding of the interaction of individual behavior and institution in the DA, we provide a model that accounts for several important regularities of double auction data that no one of the previous models predicts or replicates in simulations. For comparison of the predictions of the last three models with properties of experimental data, see Cason and Friedman [1], [2].

We model individual behavior of sellers and buyers in a continuous DA and demonstrate that the persistent puzzle of convergence to CE prices and allocations in the DA can be resolved with traders whose information processing and strategy choices are simple and intuitive. For each possible bid each buyer forms a subjective belief that some some seller will accept her bid. The buyer then determines which bid will maximize her own expected surplus. Similarly, each seller determines which offer will maximize his expected surplus. Subjective beliefs are formed using only observed market activity, including bids, offers, and accepts of bids and offers. This procedure does not require any knowledge of the types (costs and valuations) of other buyers and sellers; in fact, traders in this model do not even have beliefs about the types of others. Nevertheless, this behavior results in efficient allocations, and convergence of transaction prices to within a few cents of CE prices within several periods of trading. In addition, these beliefs respond quickly to changes in market conditions, such as shifts in market demand or supply.

The organization of the paper is as follows. The model is formulated in Section 2. Simulations of the model are shown and some important statistical properties of these simulations are reported in Section 3.3. Section 4, the conclusion, summarizes the relationship between our model and experimental data.

2 The Model

Like most forms of market organization, the double auction is an informationally decentralized system. Our model emphasizes this structure in order to give a more compelling answer to Hayek's question: How is privately held information coordinated through the market process? Hurwicz, Radner, and Reiter [8] have shown that in general equilibrium environments, even with non-convexities, there are simple and intuitive forms of market organization and bidding behavior (which they call the B process) that lead to Pareto optimal outcomes. Gjerstad and Shachat [6] construct a map between partial equilibrium environments of standard market experiments and general equilibrium economies. In this paper, we develop a model of informationally decentralized bargaining for these environments which results not only in Pareto optimal outcomes, but also results in substantial stability of transaction prices. Our bargaining model, together with the general equilibrium interpretation of the environments considered here, results in a model of learning competitive equilibrium in a class of general equilibrium environments.

In this section, we describe the elements of a microeconomic system, interpret the double auction environment and institution within this framework, and construct an informationally decentralized model of trader behavior for these environments in the double auction institution.

The double auction is an example of a *microeconomic system* as in Hurwicz [9] and Smith [13]. The primary features of a microeconomic system are the *environment* $\mathbf{e}$, consisting of the characteristics of the economic agents, and the *institution* $\mathbf{I}$, which includes the messages that traders may send to one another, the allocation rules, and the adjustment process rules. A microeconomy is an economic system $\mathbf{S} = (\mathbf{e}, \mathbf{I})$ – together with behavioral actions β^i for market participants – as shown in figure 1.

The environment $\mathbf{e}$ consists of a set $\mathcal{A} = \{1, 2, \ldots, n\}$ of agents, and for each agent i characteristics $\mathbf{e^i}$ consisting of that agent's preferences, technology, and endowment. The environment is then $\mathbf{e} = \prod_{i \in \mathcal{A}} \mathbf{e^i}$. The institution $\mathbf{I}$ consists of a message space M^i for each agent, an adjustment process rule specifying the sequence of agent messages, and an outcome function or allocation function $h(m_t) = (h^1(m_t), h^2(m_t), \ldots, h^n(m_t))$, where $m_t = (m_t^1, m_t^2, \ldots, m_t^n) \in M_t = \prod_{i \in \mathcal{A}} M_t^i$ is the vector of agents' messages.

According to Smith ([13], p. 930)

> "We want to measure messages because we want to be able to identify the behavioral modes, $\beta^i(e^i, I)$, revealed by the agents and test hypotheses derived from theories about agent behavior."

When an environment $\mathbf{e}$ and an institution $\mathbf{I}$ are specified in a market experiment, and an outcome $\mathbf{X}$ is observed, the only elements remaining to be specified are the behavioral actions $\{\beta^i(H_t \mid \mathbf{e^i}, \mathbf{I})\}_{i \in \mathcal{A}}$, where H_t is the history of activity observed by agents through time t. The focus of the research in this paper is to specify forms of behavior $\{\beta^i(H_t \mid e^i, \mathbf{I})\}_{i \in \mathcal{A}}$ that are consistent with observations $\mathbf{X}$ from exchange environments $\mathbf{e}$ when the institution $\mathbf{I}$ is the double auction. We now describe a representation of the double auction in terms of this framework.

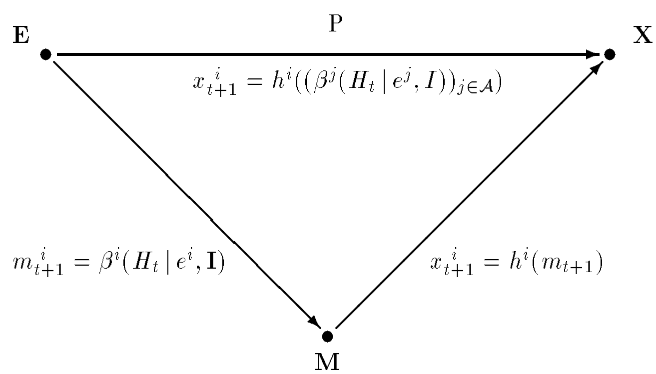

Figure 1: A microeconomic system.

2.1 Environment

In the double auction market environments we consider, there are two goods: an experimental currency X and a fictitious commodity Y. Our theory addresses the case of markets with the set of traders partitioned into a group I of sellers and a group J of buyers. Both types of agents (sellers and buyers) are assumed to have preferences over monetary rewards that are monotonically increasing.

Seller $i \in I$ has a vector of induced unit costs $c_i = (c_i^1, c_i^2, \ldots, c_i^{m_i})$ for production of an abstract commodity. Here c_i^1 is the cost to seller i of the first unit sold, c_i^2 the cost of the second unit, and so on. We index the cost of each unit because the model of trader behavior is developed for traders who frequently sell (or purchase) multiple units. The gain to seller i on his k^{th} unit sold is the difference $\pi_{s,i}^k(p_k, c_i^k) = p_k - c_i^k$ between the price p_k received from a buyer for that unit, and the cost c_i^k at which the unit is produced. If seller i sells $\mu_i \leq m_i$ units at prices $p_1, p_2, \ldots, p_{\mu_i}$, then the utility to this seller is $U_{s,i}\left(\sum_{k=1}^{\mu_i}(p_k - c_i^k)\right)$, where $U_{s,i}(\cdot)$ is monotonically increasing.[1]

Gjerstad and Shachat [6] show that the cost vector of each seller $i \in I$ is dual to a technology which is described by a production function $f_i(x)$. Let $\bar{x}_{s,i} \geq \sum_{\iota=1}^{m_i} c_i^\iota$. A seller with an endowment $\omega_{s,i} = (\bar{x}_{s,i}, 0)$ will have sufficient

[1] In Section 2.4.7, where sellers' strategies are formulated, we assume that seller i attempts to maximize surplus on each of his m_i units separately and in sequence.

currency (the input good) to produce each of the m_i units for which he has finite cost. Then characteristics of seller i are described by the vector $\mathbf{e^{s,i}} = (f_i, \omega_{s,i})$. In example 1 below, we carry out this construction for one seller in a market experiment.

Buyer $j \in J$ has a vector of unit valuations $v_j = (v_j^1, v_j^2, \ldots, v_j^{n_j})$, where v_j^1 is the redemption value for the first unit acquired, v_j^2 is the redemption value for the second, and so forth. Buyer j has an endowment $\bar{x}_{b,j}$ of trading currency that is sufficient to purchase each unit at a price up to the redemption value of the unit, i.e., $\bar{x}_{b,j} \geq \sum_{l=1}^{n_j} v_j^l$. Monetary rewards for buyers are the difference $\pi_{b,j}^l(p_l, v_j^l) = v_j^l - p_l$ between the redemption values of units purchased and the price p_l paid to a seller. If buyer j purchases $\nu_j \leq n_j$ units at prices $p_1, p_2, \ldots, p_{\nu_j}$ the monetary gain from trading for buyer j is $\sum_{l=1}^{\nu_j} (v_j^l - p_l)$, and the utility of this monetary gain is $U_{b,j}\left(\sum_{l=1}^{\nu_j} (v_j^l - p_l)\right)$, where $U_{b,j}(\cdot)$ is monotonically increasing.

For any vector of valuations v_j, Gjerstad and Shachat [6] show that there is a quasi-linear utility function $u_j(x, y) = x + v_j(y) - \bar{x}_j$ and an endowment $\omega_j = (\bar{x}_j, 0)$ such that the demand for good X by this buyer is v_j. The characteristics of buyer j are then $\mathbf{e^{b,j}} = (u_j, \omega_{b,j})$. In example 1, we carry out their construction for one buyer in a market experiment.

We describe the environment of an induced cost and valuation experiment by the collection

$$\mathbf{e} = \{(f_i, \omega_{s,i})\}_{i \in I} \cup \{(u_j, \omega_{b,j})\}_{j \in J}.$$

Example 1 Figure 2 shows supply and demand conditions for market trading experiment 3pda01 run in the experimental lab at the University of Arizona by Vernon Smith and Arlington Williams. In this market there are four buyers, each with positive valuations for three units, and four sellers, each with finite costs for three units. The vector of buyers' valuations is

$$v = \{\{3.30, 2.25, 2.10\}, \{2.80, 2.35, 2.20\}, \{2.60, 2.40, 2.15\}, \{3.05, 2.35, 2.30\}\}.$$

The vector of sellers' costs is

$$c = \{\{1.90, 2.35, 2.50\}, \{1.40, 2.45, 2.60\}, \{2.10, 2.30, 2.55\}, \{1.65, 2.35, 2.40\}\}.$$

Since buyer j with redemption value v_j^l makes a monetary gain at any purchase price $p < v_j^l$, and since buyers' preferences are assumed monotonically increasing in monetary gain, this buyer is willing to pay any price up to v_j^l for the l^{th} unit purchased. Therefore, the demand shown in figure 2 is determined by arraying the buyers' redemption value vectors. Supply is obtained analogously.

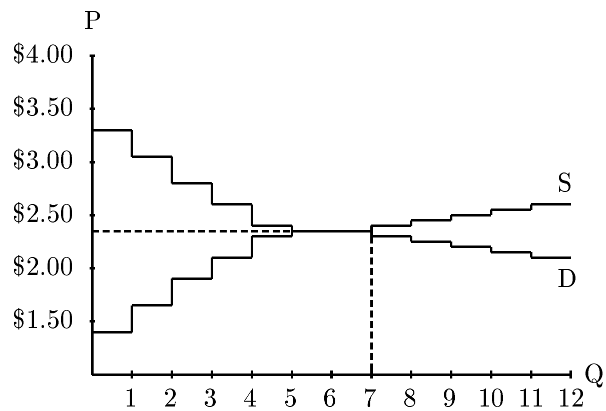

Figure 2: Supply and demand conditions for market experiment 3pda01.

The diagram on the left of figure 3 shows the production function $f_1(x)$ that is dual to the cost vector $c_1 = \{1.90, 2.35, 2.50\}$ for seller 1 in this market, where

$$f_1(x) = \begin{cases} x/1.90, & 0 \leq x \leq 1.90; \\ (x + 0.45)/2.35, & 1.90 < x \leq 4.25; \\ (x + 0.75)/2.50, & 4.25 < x \leq 6.75; \\ 3, & 6.75 < x. \end{cases} \tag{1}$$

We assume that seller 1 has an endowment $\omega_{s,1} = (\bar{x}_{s,1}, 0)$ where $\bar{x}_{s,1}$ is large enough to produce all three units. It is easy to verify that the supply of a cost minimizing producer with the production function f_1 will be 0 units if the price is below 1.90, 1 unit if the price is between 1.90 and 2.35, and so on, so the seller's vector of unit costs is dual to the production function in equation (1). The characteristics of seller 1 are then $\mathbf{e^1} = (f_1, \omega_{s,1})$.

The diagram on the right of figure 3 shows two indifference curves for buyer 1 whose vector of unit valuations is $v_1 = \{3.30, 2.25, 2.10\}$. For buyer 1, define the valuation function v_1 by

$$v_1(y) = \begin{cases} 3.30\,y, & 0 \leq y \leq 1; \\ 2.25\,y + 1.05, & 1 < y \leq 2; \\ 2.10\,y + 1.35, & 2 < y \leq 3; \\ 7.65, & 3 < y. \end{cases} \tag{2}$$

If we assume that buyer 1 has the endowment $\omega_{b,1} = (\bar{x}_{b,1}, 0) = (7.65, 0)$ and that the utility function of buyer 1 is $u_1(x, y) = x + v_1(y) - 7.65$, then the lower indifference curve in figure 3 corresponds to $u_1 = 0$ and the upper one to $u_1 = 1$.

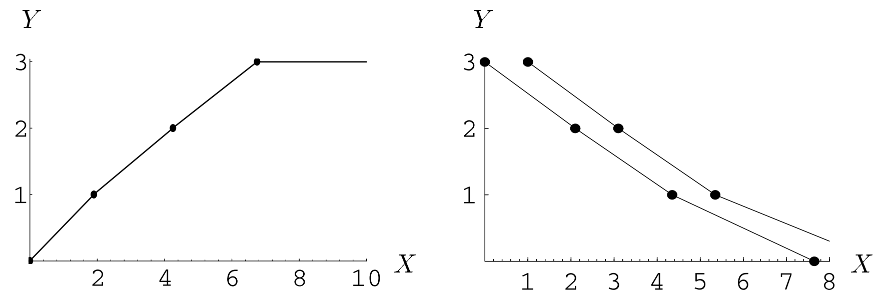

Figure 3: Production function (left) and indifference curves (right) corresponding to costs for seller 1 and values for buyer 1 in example 1.

If the production functions and endowments of the remaining 3 sellers and the utility functions and endowments of the other 3 buyer are constructed in this way, then the environment is $\mathbf{e} = \{(f_i, \omega_{s,i})\}_{i=1}^{4} \cup \{(u_j, \omega_{b,j})\}_{j=1}^{4}$. In what follows, we refer to this market as the symmetric market design.

Trading periods

A typical laboratory market experiment involves trading over several periods. Each seller has costs induced for the trading period, and each buyer has valuations induced. A buyer's valuation for a unit remains in effect throughout the trading period or until the buyer transacts that unit. After a unit is transacted, the seller's cost and the buyer's valuation for the unit just transacted are removed from the supply and demand schedules, trading continues, and this process proceeds until there are no more surplus enhancing trades remaining, or until time expires in the trading period. At the conclusion of a trading period, the costs and valuations are reinitialized – possibly at different amounts – in the subsequent period. In market experiment 3pda01 the supply and demand conditions in figure 2 were employed in each of nine trading periods, each lasting 300 seconds.

2.2 Institution

In the double auction sellers post ask prices, and buyers post bids. The message space defines the set of allowable messages for each agent. In this paper, we consider the double auction with a bid-ask spread reduction rule (defined below).

In effect, this produces restrictions on agents' messages as a function of some previous messages of other agents. In a microeconomic system, adjustment process rules specify the set of allowable messages for each trader, the time when exchange of messages begins, a transition rule governing the sequencing and exchange of messages, and a stopping rule. The DA imposes no restrictions on the sequencing of messages: any trader can send a message at any time during the trading period. Allocation of units is by mutual consent between any buyer and seller. If a seller's ask is acceptable to a buyer then a transaction is completed when the buyer takes (accepts) the seller's ask. Similarly, a buyer's bid may be accepted by a seller.

Definition 1 (Message space) Let $\tilde{N} = \{x : x = \frac{n}{100} \text{ for } n \in N\}$. Seller i at time t has a message space $M_t^{s,i}$ where $M_t^{s,i} \subset \{i\} \times \{0\} \times \tilde{N}$. Buyer j has a message space $M_t^{b,j}$ where $M_t^{b,j} \subset \{0\} \times \{j\} \times \tilde{N}$.

Definition 2 (Asks) An ask a by seller i is an amount which seller i is willing to accept from a buyer as payment for a unit of the commodity being traded. To submit an ask of a, seller i sends the message $(i, 0, a)$.

Definition 3 (Bids) A bid b by buyer j is an amount that buyer j is willing to pay to some seller for a unit. Buyer j submits this bid by sending the message $(0, j, b)$.

Definition 4 (Spread reduction rule) The lowest ask in the market at any time is called the outstanding ask and is denoted oa. At any time sellers may place an ask $a \in \tilde{N}$ with $a < oa$. The highest bid is called the outstanding bid ob, denoted ob. At any time, buyers may make a bid $b \in \tilde{N}$ above the outstanding bid. The outstanding ask oa and outstanding bid ob define the bid-ask spread $[ob, oa]$. In markets with a *spread reduction rule* all bids and asks must fall in the bid-ask spread.

Note that in a double auction with the bid-ask spread reduction rule, any ask that is permissible must be lower than the current outstanding ask, so each new ask either results in a trade or it becomes the new outstanding ask. A similar remark applies to bids.

Definition 5 (Acceptance) If seller i sends the message $(i, 0, a)$ and holds the outstanding ask $oa = a$, then a take of oa by buyer j is an agreement by j to purchase a unit from seller i at the transaction price $p = oa$. Buyer j accepts the outstanding ask oa by sending the message $(0, j, b)$ where $b \geq oa$. Similarly, if the outstanding bid ob is held by buyer j', then a take of ob by seller i' is an agreement by i' to sell a unit to buyer j' at the transaction price $p = ob$.

2.3 Observed history (Outcome)

Example 2 (Messages and histories) For the market of example 1, depicted in figure 2, if the first action in the period is an offer of 3.00

by seller 3 we indicate this with the message $m_1 = (3,\ 0,\ 3.00)$. Suppose that the next action is a bid of 3.00 by buyer 1, which we indicate with the message $m_2 = (0,\ 1,\ 3.00)$. At this point a trade is completed between seller 3 and buyer 1 at the price 3.00. We indicate the history of these two messages by the list

$$\begin{aligned} H_2 &= \{h_1, h_2\} \\ &= \{(3,\ 0,\ 3.00), (3,\ 1,\ 3.00)\}. \end{aligned}$$

Note that in H_2 the triple h_2 unambiguously denotes an accept by buyer 1 of the offer of 3.00 made with message 1 by seller 3, because seller 3 holds the outstanding offer of 3.00, which is the transaction price. (See 'Accept of oa' in definition 6.)

Definition 6 (Histories) After n messages have been sent, there is a history H_n of length n comprised of n ordered triples. For each message $m_{n+1} = (m_{n+1,1},\ m_{n+1,2},\ m_{n+1,3})$, one of six cases will hold.

Invalid ask or bid A message $m_{n+1} = (i, 0, a)$ is not valid if $a \geq oa$. An invalid ask will not be included in the history. In effect, the institution ignores messages that violate the spread reduction rule. Similarly, a message $m_{n+1} = (0, j, b)$ is not valid if $b \leq ob$.

No ask outstanding If no ask has been made since the last transaction, then there is no outstanding ask, and any ask $a \in \tilde{N}$ is valid. If in addition $m_{n+1,\,3} > ob$, then $h_{n+1} = m_{n+1}$.

No bid outstanding Similarly, if no bid has been made since the last transaction, then there is no outstanding bid, and any bid $b \in \tilde{N}$ is valid. If $m_{n+1,\,3} < oa$, then $h_{n+1} = m_{n+1}$.

Accept of *ob* If $m_{n+1,1} \neq 0$ and $m_{n+1,3} \leq ob$ then seller $m_{n+1,1}$ is making an offer at or below ob, so m_{n+1} is an accept of ob. The buyer's identity is found by looking back in H_n and finding the last h_k with $h_{k,2} \neq 0$, that is $k^* = max\{k : h_{k,\,2} \neq 0\}$. Then $(h_{n+1,1},\ h_{n+1,2},\ h_{n+1,3}) = (m_{n+1,1},\ h_{k^*,2},\ ob)$.

Accept of *oa* If $m_{n+1,2} \neq 0$ and $m_{n+1,3} \geq oa$ then m_{n+1} is an accept of oa (by buyer $m_{n+2,2}$). The seller's identity is found by looking back in H_n and finding $k^* = max\{k : h_{k,\,1} \neq 0\}$. Then $(h_{n+1,1},\ h_{n+1,2},\ h_{n+1,3}) = (h_{k^*,1},\ m_{n+1,2},\ oa)$.

Improving ask or bid If $m_{n+1,3} \in (ob, oa)$ then m_{n+1} is either an improving ask, or an improving bid, and $h_{n+1} = m_{n+1}$.

2.4 Behavior

2.4.1 Frequencies of takes

As noted in Section 2.1, sellers attempt to maximize $\pi_{s,i}^{k}(p_k,\ c_i^k)$ and buyers attempt to maximize $\pi_{b,j}^{l}(p_l,\ v_j^l)$. Since asks or bids must be accepted in order to result in a transaction, we take the point of view that sellers will maximize *expected* surplus myopically, where the expectation is taken relative to beliefs $p(a)$ that an ask a will be accepted by some buyer. These beliefs are formed on the basis of observed market data (as described in Section 2.4.2). Similarly, buyers are assumed to maximize expected surplus myopically, where the expectation is taken relative to beliefs $q(b)$ that a bid b will be accepted by some seller.

When traders form their belief, the history that they consider is restricted to those messages that lead up to the last L transactions, where $L \in \{0, 1, 2, \dots\}$. The parameter L is the memory length of traders. Note that the number of messages remembered and the clock time elapsed within the traders' memory may vary, while the number of trades completed within the traders' memory does not vary (once L trades have occurred). The next definition provides a procedure for truncating the history, so that beliefs can be constructed using the data within the traders' memory. The procedure for constructing beliefs using this (truncated) history is described in Section 2.4.2.

Note 1 We will work with the vector H_n, although traders do not have access to all the information in H_n. Traders know their own asks or bids, but do not know the identity of the trader making other bids or asks. Information about identities is not used in the formation of beliefs or strategies, so use of H_n is made only to avoid complicating notation.

Definition 7 (Remembered history) Let $\mathcal{H}_n$ be the space of possible history vectors of length n. Given $H_n \in \mathcal{H}_n$, we make the following definitions.

Trade function For a vector H_n, define a function $T : \mathcal{H}_n \mapsto \{0,1\}^n$ by setting $T_k(H_n) = I_{\{h_{k,1} \cdot h_{k,2} > 0\}}(h_k)$. Then each component T_k of T indicates whether a trade occurred in the k–th element of the history.

Number of trades Let $x = (x_1, x_2, \dots, x_n)$. For each n, define $S_n : \{0,1\}^n \mapsto N$ by $S_n(x) = \sum_{k=1}^{n} x_k$. Then $S_n(T(H_n))$ is the number of trades resulting from the first n messages.

Remembered history Let L be the memory length of a given trader. For fixed n and H_n, to simplify notation, let $S = S_n(T(H_n))$. Let n' be the position of trade $S - L$ if $S > L$, and let $n' = 0$ if $S \leq L$. Define $\tilde{H}_n^{(L)}$ by $\tilde{H}_n^{(L)} = \{h_{n'+1}, h_{n'+2}, ..., h_n\}$.

Deletion of *oa* and *ob* from history Let $n'' = \max\{k : T_k(h_k) = 1\}$, i.e., n'' is the index of the most recent trade. Let n^* be the index of the lowest (most recent) ask in the vector $\{h_{n''+1}, ..., h_n\}$. Let n_* be the index of the highest bid in the vector $\{h_{n''+1}, ..., h_n\}$. Note that if $T_n(h_n) = 0$, then $h_{n,3}$ is either the outstanding ask or the outstanding bid, as a consequence of the spread reduction rule, and if $T_n(h_n) = 1$ then there is no outstanding bid and no outstanding ask. If $T_n(h_n) = 0$ and $h_{n,1} = 0$, then $h_{n,3}$ is the outstanding bid. If $h_{k,1} \neq 0$ for some $k \in \{n'' + 1, \dots, n - 1\}$, then $n^* \neq \emptyset$ and we define $H_n^{(L)}$ by $H_n^{(L)} \equiv \{h_{n'+1}, h_{n'+2}, \dots, h_{n^*-1}, h_{n^*+1}, \dots, h_{n-1}\}$. That is, $H_n^{(L)}$ is $\tilde{H}_n^{(L)}$ with

h_{n^*} and h_{n_*} removed. This is done because it is not known at time n if the outstanding ask or bid will be accepted. The other case – where $h_{n,3}$ is the outstanding ask – is treated similarly. Then $H_n^{(L)}$ is the history remembered by traders with memory length L who observe the history H_n.

Set of asks and bids Let $D_n^{(L)}$ be the set of all asks and bids that have been made in $H_n^{(L)}$, i.e., $D_n^{(L)} \equiv \bigcup_{k\in\{n'+1,\ldots,n\}\setminus\{n_*,n^*\}}\{h_{k,3}\}$.

Definition 8 (Ask frequencies) For each $d \in D_n^{(L)}$, let $A(d)$ be the total number of asks that have been made at d, and let $TA(d)$ be the total number of these that have been accepted. Let $RA(d) \equiv A(d) - TA(d)$ be the rejected asks at d.

For $A(d)$, the counting procedure is as follows. For each $k \in \{n' + 1, ..., n\} \setminus \{n_*, n^*\}$, if $h_{k,3} = d$, $h_{k,1} \neq 0$ and $h_{k,2} = 0$, then $A(d)$ is incremented by one. If $h_{k,3} = d$ and $T_k(h_k) = 1$, then h_k is either a taken ask or a taken bid. To determine which is the case, find $m^* = \min\{m \geq 1 : h_{k-m,3} = h_{k,3}\}$. If $h_{k-m^*,1} \neq 0$, then $A(d)$ and $TA(d)$ are incremented by one. The rejected asks at d are given by $RA(d) \equiv A(d) - TA(d)$.

Definition 9 (Bid frequencies) For each $d \in D_n^{(L)}$, let $B(d)$ be the total number of bids that have been made at d, and let $TB(d)$ be the total number of these that have been accepted. Let $RB(d) \equiv B(d) - TB(d)$. The interpretations and counting procedures for $B(d)$, $TB(d)$, and $RB(d)$ are analogous to those described in definition 8 for asks.

At each time during a market, the proportion of asks at $a \in D$ that have been accepted is

$$\check{p}(a) = \frac{TA(a)}{A(a)}$$

whenever $A(a) > 0$. The proportion of bids at $b \in D$ that have been accepted is

$$\check{q}(b) = \frac{TB(b)}{B(b)}$$

whenever $B(b) > 0$.

In stationary market environments these empirical frequencies show substantial regularity: $\check{p}(a)$ tends to be a decreasing function of a and $\check{q}(b)$ tends to be an increasing function of b.

Note 2 In what follows, the sets of asks and bids is frequently denoted D, with the subscripts and superscripts omitted. When traders have finite memory, that will be noted. After n messages have been sent, the relevant set of asks and bids is $D_n^{(L)}$ and the relevant history is $H_n^{(L)}$.

2.4.2 Beliefs

While the frequencies $\check{p}(a)$ and $\check{q}(b)$ tend to be monotonic when the number of asks and bids is large, there is more variability in small samples. For this reason, it is useful to work with a modification of these summary statistics.

Modification of $\check{p}(a)$ is made by taking the point of view that if an ask $a' < a$ is rejected then had that ask been made at a it would also have been rejected. This assumption is made because $a > a'$ and is therefore less appealing to buyers than a', which was rejected. Similarly, if ask $a' > a$ was made and taken, then that ask would also have been taken if it were made at a. Also, if a bid $b' > a$ is made, then an ask $a' = b'$ would have been taken if it had been made (the assumption being that this ask of a' would be acceptable to the buyer who bid b'). This heuristic – and an analogous one for buyers' beliefs – are formalized in the next two definitions.

Definition 10 (Sellers' beliefs) For each potential ask $a \in D$, define

$$\hat{p}(a) = \frac{\sum_{d \geq a} TA(d) + \sum_{d \geq a} B(d)}{\sum_{d \geq a} TA(d) + \sum_{d \geq a} B(d) + \sum_{d \leq a} RA(d)}. \tag{3}$$

Then $\hat{p}(a)$ is the seller's belief that an ask amount a will be acceptable to some buyer. We assume that sellers always believe that an ask at $a = 0.00$ will be accepted with certainty, and that there is some value $M > 0$ such that $\hat{p}(M) = 0$.

The notation of equation (3) is simplified by the following definitions. Let $TAG(a) = \sum_{d \geq a} TA(d)$, $BG(a) = \sum_{d \geq a} B(d)$, and $RAL(a) = \sum_{d \leq a} RA(d)$. These are the taken asks greater than or equal to a, the bids greater than or equal to a, and the rejected asks less than or equal to a, respectively. Then equation (3) may be rewritten as

$$\hat{p}(a) = \frac{TAG(a) + BG(a)}{TAG(a) + BG(a) + RAL(a)}. \tag{4}$$

Definition 11 (Buyers' beliefs) For each possible bid $b \in D$, define

$$\hat{q}(b) = \frac{\sum_{d \leq b} TB(d) + \sum_{d \leq b} A(d)}{\sum_{d \leq b} TB(d) + \sum_{d \leq b} A(d) + \sum_{d \geq b} RB(d)}. \tag{5}$$

We assume that buyers always believe that $\hat{q}(0.00) = 0$ and that there is some value $M > 0$ such that $\hat{q}(M) = 1$.

As in definition 10, to simplify notation in equation (5) we introduce functions $TBL(b) = \sum_{d \leq b} TB(d)$, $AL(b) = \sum_{d \leq b} A(d)$, and $RBG(b) = \sum_{d \geq b} RB(d)$. These are the taken bids less than or equal to b, the asks less than or equal to b, and the rejected bids greater than or equal to b. Then

$$\hat{q}(b) = \frac{TBL(b) + AL(b)}{TBL(b) + AL(b) + RBG(b)}. \tag{6}$$

With the specification of seller's beliefs in definition 10, the belief function $\hat{p}(a)$ is a monotonically decreasing function of a (proposition 1). The argument of $p(\cdot)$ is the price that the seller asks, and the value of $p(\cdot)$ at a represents the seller's assessment of the probability (belief) that an offer at a will be accepted by some buyer. It is reasonable to expect that seller's beliefs are monotonic: this captures the intuition that a trader who has seen an ask of a rejected should decrease the belief that a will be accepted later, and decrease the belief that an ask at any value greater than a will be accepted. The buyers' belief function has an analogous property: $\hat{q}(b)$ is a monotonically increasing function of the bid b.

2.4.3 Spread reduction rule and beliefs

The spread reduction rule has the effect of making the probability of a take for an ask $a \geq oa$ equal to 0 (where oa is the outstanding offer from definition 4). We denote this modification of $\hat{p}(a)$ by $\tilde{p}(a)$, where $\tilde{p}(a) = \hat{p}(a)$ if $a < oa$, and $\tilde{p}(a) = 0$ if $a \geq oa$. Similarly, $\tilde{q}(b) = 0$ for all b with $b \leq ob$. These facts are incorporated into traders' beliefs in the following definition.

Definition 12 Let $\tilde{p}(a) = \hat{p}(a) \cdot I_{[0,oa)}(a)$ for each $a \in D$. That is $\tilde{p}(a) = \hat{p}(a)$ if $a < oa$ and $\hat{p}(a) = 0$ if $a \geq oa$. For all $b \in D$, let $\tilde{q}(b) = \hat{q}(b) \cdot I_{(ob,M]}(b)$.

2.4.4 Cubic spline interpolation

The belief functions in definition 12 are defined on the set D of all offers and bids within the trader's memory. These beliefs are extended to the positive reals using cubic spline interpolation. For each successive pair of data points $(a_k, \tilde{p}(a_k))$ and $(a_{k+1}, \tilde{p}(a_{k+1}))$, we construct a cubic equation $p(a) = \alpha_3 a^3 + \alpha_2 a^2 + \alpha_1 a + \alpha_0$ passing through these two points with the following four properties:

1. $p(a_k) = \tilde{p}(a_k)$;
2. $p(a_{k+1}) = \tilde{p}(a_{k+1})$;
3. $p'(a_k) = 0$;
4. $p'(a_{k+1}) = 0$.

These four conditions generate the four equations represented in matrix equation (7) below. The coefficients α_j are obtained as the solution to the equation

$$\begin{bmatrix} a_k^3 & a_k^2 & a_k & 1 \\ a_{k+1}^3 & a_{k+1}^2 & a_{k+1} & 1 \\ 3\,a_k^2 & 2\,a_k & 1 & 0 \\ 3\,a_{k+1}^2 & 2\,a_{k+1} & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} \tilde{p}(a_k) \\ \tilde{p}(a_{k+1}) \\ 0 \\ 0 \end{bmatrix}. \qquad (7)$$

The function $q(b)$ is defined similarly from $(b_k, \tilde{q}(b_k))$ and $(b_{k+1}, \tilde{q}(b_{k+1}))$.

2.4.5 Monotonicity of beliefs

The function $\hat{p}(a)$ defined in Section 2.4.2 is monotonically non-increasing. That is, as the ask a is increased, $\hat{p}(a)$ – the belief that an ask a will be accepted – is non-increasing in a. Similarly, $\hat{q}(b)$ is non-decreasing in b: buyers believe higher bids are more likely to be accepted. These results are proven in propositions 1 and 2. This monotonicity property is extended successively to $\tilde{p}(a)$, $\tilde{q}(b)$, $p(a)$, and $q(b)$ in propositions 3 – 6.

Proposition 1 For all $a_1 \in D$, $a_2 \in D$, with $a_1 < a_2$, $\hat{p}(a_1) \geq \hat{p}(a_2)$.

Proof Let $G(a) = TAG(a) + BG(a)$. Note that $G(a_2) \leq G(a_1)$ and $RAL(a_1) \leq RAL(a_2)$ because $a_1 < a_2$. Multiplying these two inequalities results in

$$G(a_2)\,RAL(a_1) \leq G(a_1)\,RAL(a_2). \tag{8}$$

Now add $G(a_2)\,G(a_1)$ to both sides of inequality (8) and from this sum factor out $G(a_2)$ from the left side of the equation and factor $G(a_1)$ out of the right side, then divide both sides of the resulting inequality by $[G(a_1) + RAL(a_1)]\,[G(a_2) + RAL(a_2)]$ to get $\hat{p}(a_1) \geq \hat{p}(a_2)$. ■

Proposition 2 For all $b_1 \in D$, $b_2 \in D$, with $b_1 < b_2$, $\hat{q}(b_1) \leq \hat{q}(b_2)$.

Proof The proof is similar to the proof of proposition 1. ■

Proposition 3 The functions $\tilde{p}(a)$ is non-increasing.

Proof Since $\tilde{p}(a) = \hat{p}(a)\,I_{[0,oa)}(a)$, and since $\hat{p}(a)$ is non-increasing, $\tilde{p}(a)$ is also non-increasing. ■

Proposition 4 The functions $\tilde{q}(b)$ is non-decreasing.

Proof The proof is similar to the proof of proposition 3. ■

Proposition 5 Let $a_1 \in D$ and $a_2 \in D$, with $a_1 < a_2$. Then $p(a)$ is non-increasing on (a_1, a_2).

Proof The belief function $p(a)$ is given by $p(a) = \alpha_3\,a^3 + \alpha_2\,a^2 + \alpha_1\,a + \alpha_0$ on the interval (a_1, a_2), where the coefficients α_j are given by the solution to equation (7). The slope of $p(a)$ on this interval is

$$p'(a) = 3\,\alpha_3\,a^2 + 2\,\alpha_2\,a + \alpha_1. \tag{9}$$

The coefficients α_3, α_2, and α_1 can be obtained from equation (7) by Cramer's rule. Substitution of these values into equation (9) results in

$$p'(a) = \frac{6\,(a_2 - a_1)\,(p(a_1) - p(a_2))}{(a_2 - a_1)^4}\,(a - a_1)\,(a - a_2).$$

Note that $a_2 - a_1$, $p(a_1) - p(a_2)$, and $(a_2 - a_1)^4$ are all non-negative. Since $a - a_1 > 0$ and $a - a_2 < 0$ on (a_1, a_2), it follows that $p(a)$ is non-increasing on (a_1, a_2). ■

Proposition 6 The belief function $q(b)$ is non-decreasing.

Proof The proof is similar to the proof of proposition 5. ■

2.4.6 Monotonicity of optimal ask in cost and optimal bid in valuation

The expected surplus maximizing ask is monotonically increasing in the cost parameter c. The following example illustrates this idea, and the proposition following the example proves this property.

Example 3 Assume the beliefs of a take given ask a are:

$$p(a) = \begin{cases} 1.0, & a = 10, \\ 0.8, & a = 11, \\ 0.6, & a = 12, \\ 0.3, & a = 13. \end{cases}$$

If sellers in a DA market have costs c = 5, c = 6, ..., c = 11, there will be expected surplus functions for each of these sellers as shown in table 1. In table 1, the expected surplus maximizing asks are underlined and exhibit the non-decreasing property of the optimal ask as cost increases, for fixed beliefs $p(a)$.

Table 1. Expected surpluses for traders with beliefs in example 3.

		Cost						
		5	6	7	8	9	10	11
	10	$\underline{5.0}$	$\underline{4.0}$	3.0	2.0	1.0	0.0	—
Ask	11	4.8	$\underline{4.0}$	$\underline{3.2}$	$\underline{2.4}$	1.6	0.8	0.0
	12	4.2	3.6	3.0	$\underline{2.4}$	$\underline{1.8}$	$\underline{1.2}$	$\underline{0.6}$
	13	2.4	2.1	1.8	1.5	1.2	0.9	$\underline{0.6}$

The following proposition shows that the optimal ask for a seller when beliefs are $p(a)$ is non-decreasing in the cost parameter. An analogous result is stated for the relation between buyers' valuations and their optimal bids.

Proposition 7 Let c_1 and c_2 be cost parameters with $c_1 < c_2$, and let the ask values $\{a_1^1, a_1^2, \ldots, a_1^m\}$, $\{a_2^1, a_2^2, \ldots, a_2^n\}$ be the sets of maximizers of $\pi(p, c_1)$ and $\pi(p, c_2)$, respectively. Let

$$a_1 = \max\{a_1^1, a_1^2, \ldots, a_1^m\}$$

and let

$$a_2 = \min\{a_2^1, a_2^2, \ldots, a_2^n\}.$$

Then $a_1 \leq a_2$.

Proof Assume $a_2 > a_1$. Notice that for each a_1^i that is a maximizer of $\pi(p, c_1)$,

$$p_1(a_1^i) = \frac{(a_1 - c_1) \cdot p(a_1)}{a_1^i - c_1}.$$

Similarly, for each a_2^i that maximizes $\pi(p, c_2)$,

$$p_2(a_2^i) = \frac{(a_2 - c_2) \cdot p(a_2)}{a_2^i - c_2}.$$

Extend the functions p_1 and p_2 to functions f_1 and f_2 on the sets (c_1, ∞) and (c_2, ∞) by writing

$$f_1(a) = \frac{(a_1 - c_1) \cdot p(a_1)}{a - c_1}$$

and

$$f_2(a) = \frac{(a_2 - c_2) \cdot p(a_2)}{a - c_2}.$$

Note that f_1 and f_2 are branches of hyperbolas on their respective domains. Define the point α implicitly by setting $f_1(\alpha) = f_2(\alpha)$. It is easy to see that α exists and $\alpha \in (c_2, \infty)$. Solving the equation $f_1(\alpha) = f_2(\alpha)$ for α results in

$$\alpha = c_2 + \frac{(c_2 - c_1)(a_2 - c_2)\, p(a_2)}{(a_1 - c_1)\, p(a_1) - (a_2 - c_2)\, p(a_2)}.$$

The denominator of the second term on the right is positive since $(a_1 - c_1)\, p(a_1)$ is the maximum expected surplus for a trader with cost c_1 and $(a_2 - c_2)\, p(a_2)$ is the maximum for a trader with cost c_2, where $c_2 > c_1$. The numerator is positive also, so $\alpha \in (c_2, \infty)$. (Of course, this depends on the traders each having some ask with positive expected surplus, but this is the only case of interest.)

Either $a_2 < \alpha$ or $a_2 \geq \alpha$. In the case of $a_2 < \alpha$ it will be shown that

$$(a_2 - c_1)\, p(a_2) > (a_1 - c_1)\, p(a_1).$$

Since a_1 is the largest maximizer of $\pi(p, c_1)$, and since we have assumed that $a_2 > a_1$, a_2 may not be a maximizer of $\pi(p, c_1)$. This establishes the contradiction.

For all $a \in (c_2, \alpha)$,

$$\frac{(a_2 - c_2)\, p(a_2)}{(a - c_2)} > \frac{(a_1 - c_1)\, p(a_1)}{(a - c_1)}.$$

This holds in particular for $a = a_2$:

$$\frac{(a_2 - c_2)\, p(a_2)}{(a_2 - c_2)} > \frac{(a_1 - c_1)\, p(a_1)}{(a_2 - c_1)}$$

or

$$p(a_2) > \frac{(a_1 - c_1)\, p(a_1)}{(a_2 - c_1)}.$$

Now multiply each side in the above inequality by $(a_2 - c_1)$ to get

$$(a_2 - c_1)\, p(a_2) > (a_1 - c_1)\, p(a_1).$$

This contradicts the fact that a_1 is expected surplus maximizing for a seller with cost c_1, so $a_2 \geq \alpha$.

A similar argument shows that $a_1 \leq \alpha$.

Combining these two inequalities gives the desired result:

$$a_1 \leq \alpha \leq a_2. \blacksquare$$

Proposition 8 Let v_1 and v_2 be valuations with $v_1 < v_2$. Suppose that the bid values $\{b_1^1, b_1^2, \ldots, b_1^m\}$ is the set of maximizers of $\pi(p,\, v_1)$ and let $\{b_2^1, b_2^2, \ldots, b_2^n\}$ be the set of maximizers of $\pi(p,\, v_2)$. Let

$$b_1 = \max\{b_1^1, b_1^2, \ldots, b_1^m\}$$

and let

$$b_2 = \min\{a_2^1, b_2^2, \ldots, b_2^n\}.$$

Then $b_1 \leq b_2$.

Proof The proof is similar to the proof of the previous proposition. $\blacksquare$

2.4.7 Expected surplus maximization

When attempting to sell his k^{th} unit, seller i with cost $c_i^k < oa$ may make an offer $a \in [0,\, oa)$ which results in expected surplus $E[\pi_{s,i}^{\,k}(a,\, c_i^k)] = (a - c_i^k) \cdot p(a)$. The maximum expected surplus of seller i for the sale of this unit[2] is

$$S_{s,i}^{\,k} = \max\{ \max_{a \in (ob,\, oa)} E[\pi_{s,i}^{\,k}(a,\, c_i^k)], 0\}. \tag{10}$$

Similarly, if buyer j with valuation v_j^l for her l^{th} unit bids $b \in (ob,\, \infty)$, this results in expected surplus $E[\pi_{b,j}^{\,l}(b,\, v_j^l)] = (v_j^l - b) \cdot q(b)$. The maximum expected surplus on this unit for buyer j is

$$S_{b,j}^{\,l} = \max\{ \max_{a \in (ob,\, oa)} E[\pi_{b,j}^{\,l}(b,\, v_j^l)], 0\}. \tag{11}$$

[2]Seller i with cost vector $c_i = \{c_i^1,\, c_i^2,\, \ldots,\, c_i^{m_i}\}$ faces the problem of choosing a sequence of asks or accepts to maximize $\sum_{k=1}^{m_i} (p_i^k - c_i^k)$, where p_i^k is the purchase price received for unit k. We assume that the seller will attempt to maximize the surplus of each unit in sequence, independently of other units. In addition to simplifying the strategy choice, this is consistent with the myopic formulation of strategy choice. A similar remark applies to buyers.

2.4.8 Timing of messages

Let t be the parameter for time within a trading period. Let T be the length of the trading period and let $t_\kappa \in [0, T)$ be the time of of the κ^{th} offer, bid, or acceptance of an offer or bid. At time t_κ let $T^\kappa_{s,i}$ be the random variable that specifies the time which seller i would allow to elapse before sending a message; let $T^\kappa_{b,j}$ be the random variable that specifies the time which buyer j would allow to elapse before sending a message.

We assume that $T^\kappa_{s,i}$ is exponentially distributed, and that the parameter $\alpha_{s,i}$ in the distribution of $T^\kappa_{s,i}$ depends only on the maximum expected surplus $S^k_{s,i}$ of seller i (from equation (10)) of seller i, on the length T of the trading period, and on t_κ, the time elapsed in the trading period. We write this dependence as $\alpha_{s,i} = f_{s,i}(S^k_{s,i}; t_\kappa, T)$. Similarly for buyers $\beta_{b,j} = f_{b,j}(S^l_{b,j}; t_\kappa, T)$. Then the probability that seller i' will be the next trader to send a message in the market is

$$\begin{aligned} p_{s,i'} &= \frac{f_{s,i'}(S^k_{s,i'}; t_\kappa, T)}{\sum_{i\in I} f_{s,i}(S^k_{s,i}; t_\kappa, T) + \sum_{j\in J} f_{b,j}(S^l_{b,j}; t_\kappa, T)} \\ &= \frac{\alpha_{s,i'}}{\sum_{i\in I}\alpha_{s,i} + \sum_{j\in J}\beta_{b,j}}. \end{aligned} \tag{12}$$

Equation (12) indicates that the probability that seller i' is the next to send a message is equal to the parameter $\alpha_{s,i'}$ of seller i' divided by the sum of the parameters of all agents. This is shown in the following proposition.

Proposition 9 If $T^\kappa_{s,i}$ and $T^\kappa_{b,j}$ are independent exponentially distributed random variables on $[0,\infty)$ with parameters $\alpha_{s,i} = f_{s,i}(S^k_{s,i}; t_\kappa, T)$ and $\beta_{b,j} = f_{b,j}(S^l_{b,j}; t_\kappa, T)$, i.e.,

$$Pr\{T_{s,i} < t\} = 1 - e^{-\alpha_{s,i}\cdot t},$$

then the probability that seller i' will be the next trader to send a message is $p_{s,i'}$, where $p_{s,i'}$ is given by equation (12).

Proof Consider, for example, seller 1. Let

$$T_{s,-1} = \min_{i>1, j\geq 1}\{T_{s,i}, T_{b,j}\}.$$

This random variable is exponentially distributed with parameter

$$\alpha_{s,-1} = \sum_{i>1}\alpha_{s,i} + \sum_{j\geq 1}\beta_{b,j}.$$

Then the probability that seller 1 is the next seller to move will be the probability that $T_{s,1} < T_{s,-1}$, i.e.,

$$\begin{aligned} p_{s,1} &= Pr\{T_{s,1} < T_{s,-1}\} \\ &= \int_0^\infty \int_0^u \alpha_{s,1}\cdot e^{-\alpha_{s,1}\cdot t}\cdot \alpha_{s,-1}\cdot e^{-\alpha_{s,-1}\cdot u}\, dt\, du \\ &= \frac{\alpha_{s,1}}{\alpha_{s,1} + \alpha_{s,-1}}. \end{aligned} \tag{13}$$

After substitution of the definitions of $\alpha_{s,1}$ and $\alpha_{s,-1}$, the expression in equation (13) is the same as equation (12) for $i' = 1$. ■

There are two reasons for defining the timing of each trader's message as an exponential random variable. The first is an important conceptual issue: with this formulation, the mechanism is informationally decentralized, in that the information about each trader's surplus is not held by any agent. Each trader's timing decision is independent of any (unobserved) characteristics – such as costs or valuations – of other traders. The second issue is empirical. With this formulation, the timing of bids and asks is testable within the model, and it is possible to compare the timing data for various specifications of the functions $f_{s,i}(S^k_{s,i}; t_\kappa, T)$ and $f_{b,j}(S^l_{b,j}; t_k, T)$ with timing data from experiments. It should be noted that the arguments of $f_{s,i}(\cdot)$ may be any data observed by or known to seller i. In the formulation above, the surplus $S^k_{s,i}$ is a summary statistic derived from the information privately held by and publicly observed by seller i.

The specifications of $f_{s,i}(S^k_{s,i}; t_\kappa, T)$ and $f_{b,j}(S^l_{b,j}; t_\kappa, T)$ used in the simulations reported in Section 3 are

$$f_{s,i}(S^k_{s,i}; t_\kappa, T) = S^k_{s,i} \cdot \frac{T}{(T - a\, t_\kappa)} \tag{14}$$

and

$$f_{b,j}(S^l_{b,j}; t_\kappa, T) = S^l_{b,j} \cdot \frac{T}{(T - a\, t_\kappa)} \tag{15}$$

where $a \in (0,\, 1)$.

This specification has been chosen to reflect two empirical observations about timing of bids and offers in experimental markets. There is strong positive rank-order correlation between buyers' valuations and the order that buyers purchase units, and strong negative rank-order correlation between sellers' costs and the order that units are sold. In the formulation above, buyers with high valuations will have higher maximum expected surplus, and will therefore have a larger parameter $\beta_{b,j}$ for the timing decision. Since the expected time until buyer j sends a message is proportional to the reciprocal of $\beta_{b,j}$, buyers with high valuations will tend to send messages more frequently and will trade earlier. Similarly, sellers with low cost will trade earlier. The second observation is that trading activity is typically concentrated at the beginning of the trading period, when many high surplus units are traded, and toward the end of the period. The term $\frac{T}{(T - a\, t_\kappa)}$ in equations (14) and (15) is consistent with these observations, since high surplus units will trade earlier, but as $t_\kappa \to T$, this term approaches $\frac{1}{(1-a)}$ and low surplus units will be traded toward the close of each period if a is near 1.

Though the model is formulated so that timing data can be obtained and examined, we have followed the reduced form in equation (12) in simulations and generated messages without the time stamp.

Example 4 (Beliefs, surplus, and timing) Consider again the market of example 1. In example 2 we discuss the first two messages sent in experiment 3pda01, which result in the history $H_2 = \{(3,\, 0,\, 3.00), (3,\, 1,\, 3.00)\}$.

The set of bids and offers D where the beliefs $\hat{p}(a)$ and $\hat{q}(b)$ are calculated is $D = \{3.00\} \cup \{0.00, 10.00\}$. The values of $\hat{p}(a)$ at these three points are $\hat{p}(0.00) = 1.0$, $\hat{p}(3.00) = 1.0$, and $\hat{p}(10.00) = 0.0$. The values of $\hat{q}(b)$ at these 3 points are $\hat{q}(0.00) = 0.0$, $\hat{q}(3.00) = 1.0$, and $\hat{q}(10.00) = 1.0$. Since there is no outstanding ask or bid, the spread reduction rule has no effect, so $\tilde{p}(a) = \hat{p}(a)$ for all $a \in D$ and $\tilde{q}(b) = \hat{q}(b)$ for all $b \in D$. Finally, the belief functions $p(a)$ and $q(b)$, shown in figure 4, are

$$p(a) = \begin{cases} 1, & 0 \leq b \leq 3.00; \\ \frac{1}{343}\left(100 + 180x - 39x^2 + 2x^3\right), & 3.00 \leq b \leq 10.00; \\ 0, & 10.00 \leq b, \end{cases} \tag{16}$$

and

$$q(b) = \begin{cases} \frac{1}{27}\, x^2\,(3 - 2\,x), & 0.00 \leq b \leq 3.00; \\ 1, & b > 3.00. \end{cases} \tag{17}$$

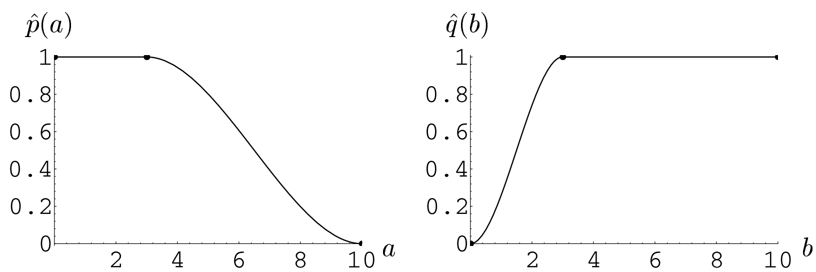

Figure 4: Sellers' beliefs (left) and buyers' beliefs (right) after trade in example 3.

After the trade between seller 3 and buyer 1 in this example the lowest cost units of sellers 1 – 4 are 1.90, 1.40, 2.30, and 1.65, respectively. The highest unit values for buyers 1 – 4 are 2.25, 2.80, 2.60, and 3.05. The expected surplus maximizing bids and offers and the maximum expected surplus for each agent is easily determined from these values and costs and the belief functions in equations (16) and (17). The values of the maximum expected surplus for buyers 1 – 4 are 0.38, 0.66, 0.55, and 0.81. For sellers 1 – 4, the values of maximum expected surplus are 2.55, 2.91, 2.27, and 2.73. As a result, given the formulation of the timing of messages in equations (14) and (15), in our model an offer will be more than 4 times as likely as a bid with this history. Even if the next message is an offer, the expected surplus of sellers will continue to be greater than expected surplus for buyers, and there is high probability of a series of decreasing offers. As a result, the next transaction price is likely to move toward the market equilibrium of 2.35.

3 Simulations

While the formation of traders' beliefs, their choices of strategies, and the timing of messages are simple and intuitive, the dynamics of the model are complex due to the non-stationarity of beliefs and the probability distribution over the timing of agents' messages. As a result, analytic characterization of properties of the model are difficult to obtain. For this reason, much of the evidence presented on performance of the model is from simulations. It may be possible to obtain analytic results on asymptotic convergence of prices to an approximate equilibrium, but asymptotic convergence alone does not provide information about the path. For the important question of the path of convergence to equilibrium, simulations are a useful tool for investigating properties of this model.

3.1 Criteria

The primary objective of this section is to demonstrate that prices and allocations in our model converge to the competitive equilibrium price and allocation. To identify the effect on convergence to competitive equilibrium of the belief formation and strategy choice defined in Section 2.4, we compare the outcomes of simulations of our model to the "Zero-intelligence trader" (ZI) model of Gode and Sunder [7], which has no belief formation or learning. [3] By convergence of the sequence of prices we mean that for some $n_0 \geq 1$, each element of the sequence $(p_n)_{n=n_0}^{N}$ of transaction prices is "close to" p_e, the competitive equilibrium price. This condition is met if the mean absolute deviation of transaction price from equilibrium price is small, so we measure convergence using this statistic. Of course, convergence to competitive equilibrium implies convergence to a Pareto optimum, so we also test this weaker condition. With efficiency measured as the ratio of surplus extracted by agents to total surplus possible, we find that in simulation of the model, market allocations are nearly efficient (over 99.9% of possible surplus is extracted after several periods of trading), and prices are close to competitive equilibrium prices. Note that while we do not examine the market allocation directly, the outcome is a competitive equilibrium if and only if the transaction prices are the competitive equilibrium price *and* the allocation is Pareto optimal, and we do establish that these two conditions hold (approximately) in simulations of our model.

3.2 Environments

Throughout this section, we consider a market design with four sellers each with finite unit costs for 3 units, and four buyers, each with positive unit valuations for 3 units. The unit costs and valuations for the design we consider are given in example 1 of Section 2.1 and are depicted in figure 2 (and also on the left

[3] In this model, sellers make offers which are random and uniformly distributed on the interval $[c, M]$, where c is the seller's cost, and M is some upper bound on their set of possible choices. Buyers make bids that are uniformly distributed on $[0, v]$, where v is the buyer's valuation.

column of figure 5). We report statistics from seven laboratory market experiments reported initially in Ketcham, Smith, and Williams [10] and statistics from 100 simulations of our model with the same environment parameters. Finally, in order to demonstrate that our model is capable of responding to shifts in market conditions, we show the results of a simulation in which the market design described above is employed through 5 periods of trading, and then for the remaining five periods of the simulation, each unit cost and unit valuation is increased to amounts 0.50 above those employed in periods 1 – 5.

3.3 Evaluation

The only free parameter in the model is the memory length of traders. We report simulations with memory length $L = 5$. For short memory lengths ($L \leq 3$) the outcomes are unstable. For long memory lengths ($L \geq 8$) the outcomes are similar to those with intermediate memory lengths ($4 \leq L \leq 7$), but computation time increases significantly. It should also be noted that beliefs change more slowly in markets with shifts in supply and demand if memory length is long, so traders with intermediate and short memory length will adapt to changes in market conditions more quickly.

Efficiency The sum of consumers' and producers' surplus provides a convenient measure of efficiency for these markets. We evaluate the surplus obtained by traders in the market divided by the maximum surplus available to determine the efficiency of trade. Table 2 summarizes efficiency statistics from the seven lab markets, from 100 simulations of our model, and from 100 simulations of the ZI model. This table shows that our model attains higher efficiency than both the laboratory experiments and the ZI model simulations.

Table 2. Efficiency statistics from simulations of models and from lab data.

Periods evaluated	Symmetric Markets	Model Simulations	ZI Model Simulations
First two periods	0.907	0.9982	0.968
Entire experiment	0.959	0.9991	0.968
Last two periods	0.970	0.9992	0.967

Convergence The diagrams in the left column of figure 5 show graphs of the supply and demand conditions for the symmetric market 3pda01. On the right side of that figure in the top row is a graph of the sequence of transaction price through the 9 periods of trading in this laboratory market. In that figure, the equilibrium price is shown as a solid line across the diagram. Prices from each of the nine trading periods are separated by a vertical line, and the number of transactions in each period is indicated at the bottom of the diagram below the vertical line that indicates the end of the trading period. A simulation of our model under the same supply and demand conditions is shown on the right side of figure 5 in the center row. Price sequences from lab experiments with this design (as in the top of figure 5) and from simulations with this market design (as in the center row of figure 5) both converge quickly to prices near the competitive equilibrium price and an equilibrium quantity of trade typically occurs in each period.

The belief functions $p(a)$ and $q(b)$ shown in figure 6 are produced using data from the the end of the second period of the simulation of figure 5 using definitions 10 and 11 of Section 2.4.2. In this graph, a seller's belief that ask a will be accepted by a buyer is shown for each ask from 2.32 to 2.38; buyers' beliefs are shown for bids from 2.30 to 2.36.[4] These belief functions are monotonic (see propositions 5 and 6), so the value of the sellers' belief is $p(a) = 1$ for all $a < 2.33$ and it is 0 for all $a > 2.38$. With this belief function, and with myopic surplus maximization, the optimal ask is approximately 2.33 for any seller with unit cost below 2.30. The buyers' belief functions in this case have a similar property: the optimal bid for a buyer with valuation of 2.40 or greater is 2.35. At the beginning of each period, the sellers' costs are 1.90, 1.40, 2.10, and 1.65 and the buyers' valuations are 3.30, 2.80, 2.60, and 3.05 so the the first action at the beginning of the third period will be either an ask of 2.33 or a bid at 2.35. We see in the center row of figure 5 that the first transaction price is 2.35 in period 3 of this simulation: the belief functions and strategy choice described frequently produce transactions at equilibrium, even from the beginning of the trading period.

The figures on the bottom row of figure 5 shows a simulation of the ZI model in the symmetric market environment. While the ZI traders attain high efficiency in this market design, that model does not result in the formation of equilibrium prices. In the ZI model, there is no belief formation process. As a result there is no convergence of transaction prices to equilibrium, as the diagram on the right side of the bottom row in figure 5 clearly shows.[5] In table 3, the mean absolute deviation of transaction price from equilibrium price is shown for 100 simulations of the ZI model in the symmetric market design.[6] This statistic is also shown for 100 simulations of our model[7] and for the seven lab markets. These data

[4]Note that the range from the lowest cost to the highest valuation in this market is 1.40 to 3.30, with an equilibrium price of 2.35; beliefs are focused in a narrow range around the equilibrium price.

[5]Gode and Sunder [7] (p. 129) argue that "By the end of a period, the price series in budget constrained ZI trader markets converges to the equilibrium level almost as precisely as the price series from human trader markets does." In the ZI simulation of figure 5, final trades in 8 of 10 periods are within 0.10 of equilibrium. We apply a definition of convergence which is more demanding and conforms more closely to the intuitive notion of convergence of market prices. We argue that prices in a stable market environment converge if after several periods, the mean deviation of all trades from equilibrium is small. By this criterion, the ZI model does not converge to equilibrium.

[6]Although the mean absolute deviation in the last two periods of the ZI model simulations is less than in the first two periods, this is not the result of convergence. The price sequence in each period constitutes a draw from the same distribution.

[7]The timing specification employed in the simulations is given in equations (14) and (15).

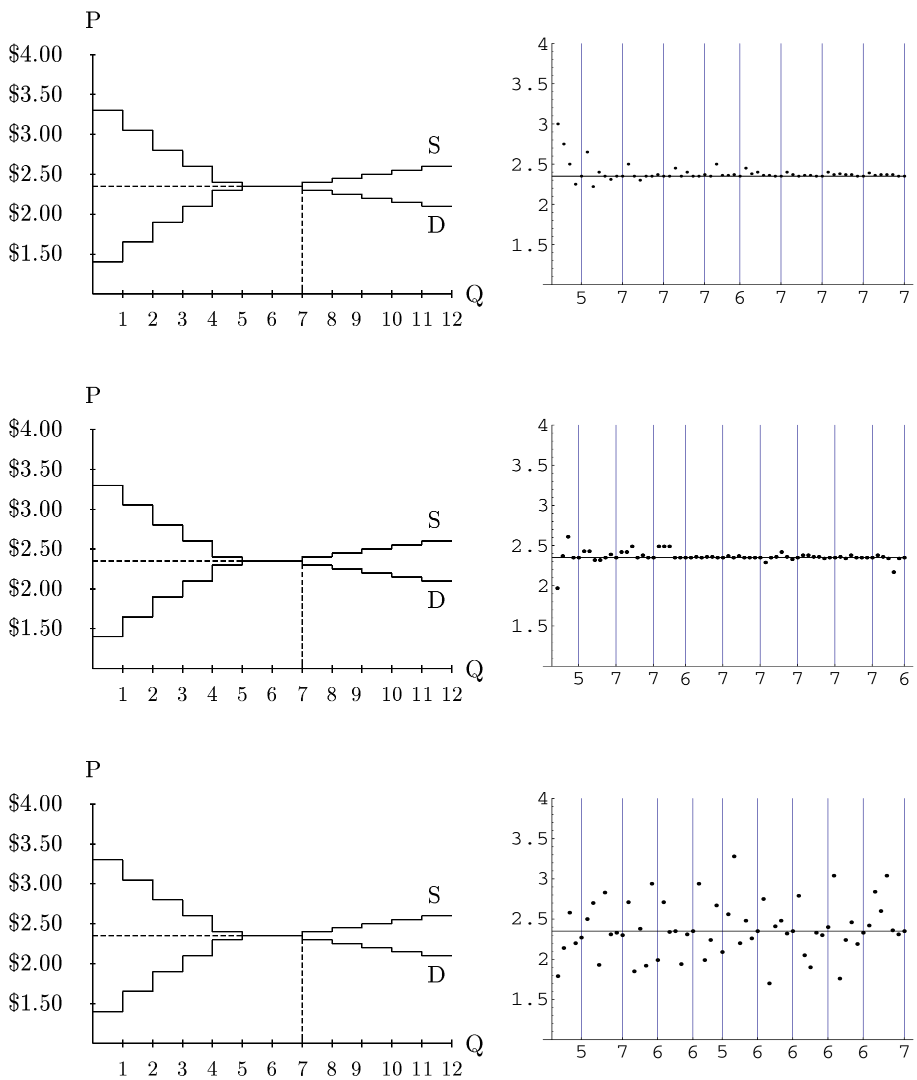

Figure 5: Supply and demand conditions (top left) and transaction prices (top right) for market experiment 3pda01; for a simulation of market 3pda01 (center) and for a ZI simulation of market 3pda01 (bottom).

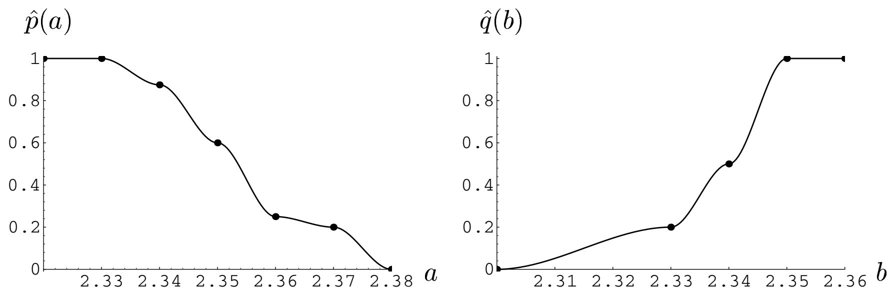

Figure 6: Sellers' beliefs (left) and buyers' beliefs (right) after five periods of trading in simulation of market 3pda01.

(and the ZI simulation graph at the bottom of figure 5) show that the ZI model does not result in convergence to competitive equilibrium. The behavior in our model does produce convergence to approximate equilibrium prices after several periods of trading. The contrast between the outcome of the simulations of the ZI model and of our model identify the effect on market convergence of belief formation and myopic surplus maximization in our model. From the graphs and the statistics, it is clear that this effect is substantial. Moreover, data on mean absolute deviation in table 3 show not only that our model converges to within a few cents of the equilibrium price, but that the rate of convergence is similar to – though initially slightly faster than – that found in laboratory experiments.

Table 3. Convergence statistics from simulations of models and from lab data.

Periods evaluated	Symmetric Markets	Model Simulations	ZI Model Simulations
First two periods	0.101	0.077	0.276
Entire experiment	0.050	0.045	0.237
Last two periods	0.022	0.040	0.209

Shifting conditions The diagram on the left of figure 7 shows two sets of supply and demand conditions. The lower set, shown with thicker lines labeled S and D, is identical to the supply and demand conditions in figure 2 and figure 5. If after several periods of trading, buyers have each valuation increased by 0.50 and sellers have the cost of each unit increased by 0.50, then the new supply and demand are those shown with thinner lines in figure 7. The equilibrium quantity of trade and the total surplus are unaffected by this shift, but the

In one alternative tested, all agents with positive surplus were equally likely to send a message. This resulted in higher variance in transaction prices and in more instability in the outcomes.

equilibrium price increases from 2.35 to 2.85. Since expectations focus near the original equilibrium after several periods of trading (see figure 6), the dynamics of movement to the new equilibrium can be examined by considering this type of market. The sequence of transaction prices from a simulation of our model in this type of market – with the shift occurring after 5 periods of trading – is shown on the right side of figure 7. From periods 6 through 10 in this market the equilibrium price is 2.85. In the simulation shown, convergence to the original equilibrium occurs by the end of period 2. Beginning in period 6, the equilibrium price shifts up 0.50. By the end of period 7 transaction prices establish near the new equilibrium price. This simulation shows that in the model developed here, traders respond to shifting market parameters, and prices quickly adjust to a new equilibrium.

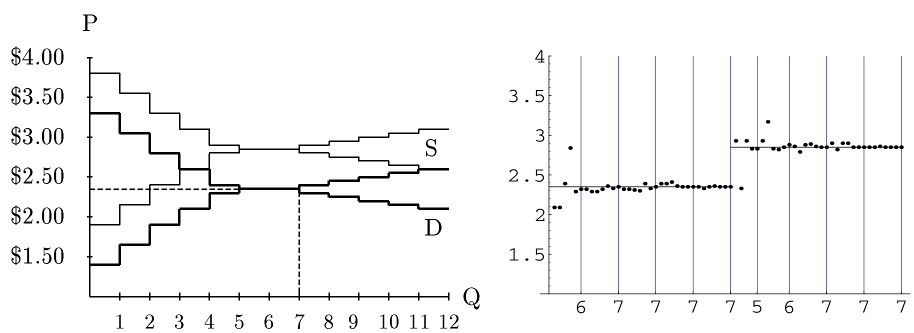

Figure 7: Supply and demand conditions and transaction prices for a simulation of a market with a supply and demand shift.

3.4 Boundaries on performance

Though we have developed a model that simultaneously converges to competitive equilibrium prices, produces efficient outcomes, and responds quickly to altered market conditions, we want to indicate direction for improvement in the outcomes of the bargaining behavior developed in our model. We do so by describing a distinctive feature of price sequences from experimental markets.

We split the sequence $(p_n)_{n=1}^N$ into two subsequences, one consisting of those exchange prices $(p_n)_{n\in N_s}$ which were initially proposed by the selling side of the market, and the other consisting of the transaction prices $(p_n)_{n\in N_b}$ which were proposed initially by the buying side. We provide evidence that in laboratory experiments, the mean of $(p_n)_{n\in N_s}$ is greater than the mean of $(p_n)_{n\in N_b}$. Yet in simulations of our model, this pattern is reversed. This observation allows us to evaluate behavior in the model and uncover a key difference between the behavior of laboratory subjects and behavior in the model, providing direction for further research on bargaining behavior in DA markets.

Table 4. Price sequence and subsequence means in symmetric markets.

Market	p_e	$\overline{p} - p_e$	$\overline{p_a} - p_e$	$\overline{p_b} - p_e$
3pda01	\$2.35	\$0.04	\$0.12	\$0.01
2pda17	\$6.20	\$0.03	\$0.07	\$0.00
2pda20	\$6.20	\$0.06	\$0.12	\$0.01
2pda21	\$5.30	−\$0.03	−\$0.01	−\$0.04
2pda24	\$4.70	\$0.02	\$0.04	−\$0.02
2pda47	\$6.35	−\$0.03	−\$0.01	−\$0.05
2pda53	\$7.55	\$0.00	\$0.01	−\$0.02
Mean	—	\$0.01	\$0.05	−\$0.02

In laboratory trading experiments, there is a clear difference between mean prices of trades initiated by sellers and those initiated by buyers. For example, in the seven symmetric markets we consider, table 4 shows the market equilibrium price for each of these experiments in column 2 and the mean difference of transaction price from the equilibrium in column 3. Column 4 shows the mean difference between the transaction price and equilibrium price for all trades initially proposed by sellers. Column 5 shows the same statistic for all trades initially proposed by buyers. Note that in each of these seven experiments, the transactions initiated by sellers have a higher mean price than those initiated by buyers. Two additional data sets – one involving over 11,000 trades – are examined in Gjerstad [5] and this is a feature of all markets in both data sets considered there. In simulations of our model we find that this feature is reversed, and as a result, we are able to discern a difference between the bargaining behavior of laboratory subjects and behavior in the model.

Consider again the belief functions $p(a)$ and $q(b)$ shown in figure 6. In the description of the beliefs in these graphs in Section 3.3, we note that at the beginning of a period, sellers in this market with these beliefs will all have an optimal offer of 2.33 and buyers' optimal bids will all be 2.35. At the beginning of the third period, the first action will be either an ask of 2.33 or a bid at 2.35. Suppose that the first action is a bid of 2.35. As a result of the spread reduction rule, buyers' bids must be greater than 2.35. A bid of 2.36 would result in expected surplus $(v_j^1 - 2.36) \cdot 1$ for each of the four buyers in this market. Since the distribution of costs and valuations at the beginning of each period in this market is symmetric, and since the probability of each trader being the next to send a message is equal to that trader's proportion of total surplus (see proposition 9), the probability that a buyer will send the next message is approximately 0.50, so that the probability of two consecutive bids is approximately 0.25, and in this case the price will be above the equilibrium price. In general, the distribution of absolute deviations from equilibrium is approximately geometric, since a low price results from a sequence of asks and a high price results from a sequence of bids. Recall that in table 4, the transactions initiated by the selling side typically are above equilibrium, so the simulations and the laboratory markets lead to the opposite result in this respect.

While this is a subtle feature of DA data, a model which eliminates this difference would capture fine aspects of bargaining behavior and would represent progress in modeling behavior in this institution.

4 Conclusions

In this paper we have defined beliefs for agents in a double auction market which are generated endogenously on the basis of observed market activity. An agent's choice of an action depends only on these beliefs and on that agent's private information about their own costs or valuations. Agents who adopt the simple bargaining strategy of myopic expected surplus maximization employing these beliefs trade at prices which converge quickly and accurately to within several cents of the market equilibrium price and reach the competitive allocation. These beliefs and strategies are flexible enough to respond quickly to changes in supply and demand conditions.

Laboratory market experiments dating back 35 years have demonstrated that human subjects quickly and reliably reach competitive equilibrium outcomes. The model developed in this paper demonstrates that this capability of double auction market participants to reach competitive equilibrium outcomes may result from simple, intuitive information processing and strategy choice. Since we know that laboratory subject operate with limited information processing capabilities and boundedly rational strategy choices, this finding resolves, at least for the class of environments considered here, the puzzle Hayek posed.

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