TWO TRIANGLES

Let \( \triangle ABC \) be the outer triangle, and \( \triangle DEF \) be the inner one, so that \( D \in BC, \ E \in AC, \ F \in AB \). Denote \( \alpha = \angle FDB, \ \beta = \angle EFA, \ \gamma = \angle DEC \), and assume without loss of generality that

(0.1) \[ \alpha \geq \beta \geq \gamma. \]

Denote \( \alpha' = \angle DFB, \ \beta' = \angle FEA, \ \gamma' = \angle EDC \). Clearly, \( \alpha' = 2\pi/3 - \beta, \ \beta' = 2\pi/3 - \gamma, \ \gamma' = 2\pi/3 - \alpha \), and hence

(0.2) \[ \beta' \geq \alpha' \geq \gamma'. \]

Consider three triangles: \( \triangle BFD, \ \triangle AEF, \ \triangle CDE \). Let us superimpose the bases \( FD, \ EF, \) and \( DE \) with a horizontal segment \( PQ \). Then the vertices \( A, \ B, \ C \) lie on a upper semicircle centered at \( Q \). Let \( R \) be the tangent point of the line through \( P \) and the semicircle; \( R \) divides the semicircle into two arcs: the front one (which can be seen from \( P \)) and the back one.

There are two cases:
1) all angles of \( \triangle ABC \) are acute;
2) one of the angles of \( \triangle ABC \) is obtuse.

In the first case, all three points \( A, \ B, \ C \) lie on the back arc. By (0.1), their clockwise order along the semicircle is \( RCAB \), which contradicts (0.2) unless \( \alpha = \beta = \gamma \).

In the second case, one point lies on the front arc. By (0.1), this point is \( C \). Denote by \( C' \) the second intersection point of the line through \( P \) and \( C \) with the semicircle; by (0.2), the clockwise order of the points along the semicircle is \( CRABC' \).

Clearly, \( \angle PCQ + \angle PC'Q = \pi \). On the other hand, points \( A \) and \( B \) lie inside the circle through \( P, \ Q, \) and \( C' \), and hence \( \angle PAQ \geq \angle PC'Q, \ \angle PBQ \geq \angle PC'Q \). Consequently, \( \angle PCQ + \angle PAQ + \angle PBQ > \pi \), a contradiction.